Some BASICs

Voltage, Current, Components and AC behavior,
Bode Plots, Transfer Functions, Thévenin Equivalent,
High-pass and Low-pass filters,…

Prefixes for Units

- For writing down small or large quantities, exponents can be used: $1.5 \times 10^6\ \Omega$, 3×10^{-9} A
- To simplify, ***prefixes*** in steps of 1000 are used:

T	Tera	$\times 10^{12}$
G	Giga	$\times 10^{9}$
M	Mega	$\times 10^{6}$
k	Kilo	$\times 10^{3}$
	1	$\times 10^{0}$
m	Milli	$\times 10^{-3}$
μ (or u)	Mikro	$\times 10^{-6}$
n	Nano	$\times 10^{-9}$
p	Piko	$\times 10^{-12}$
f	Femto	$\times 10^{-15}$
a	Atto	$\times 10^{-18}$

- Try to learn: ‘Piko × Kilo = Nano, Milli × Mega = Kilo,…’

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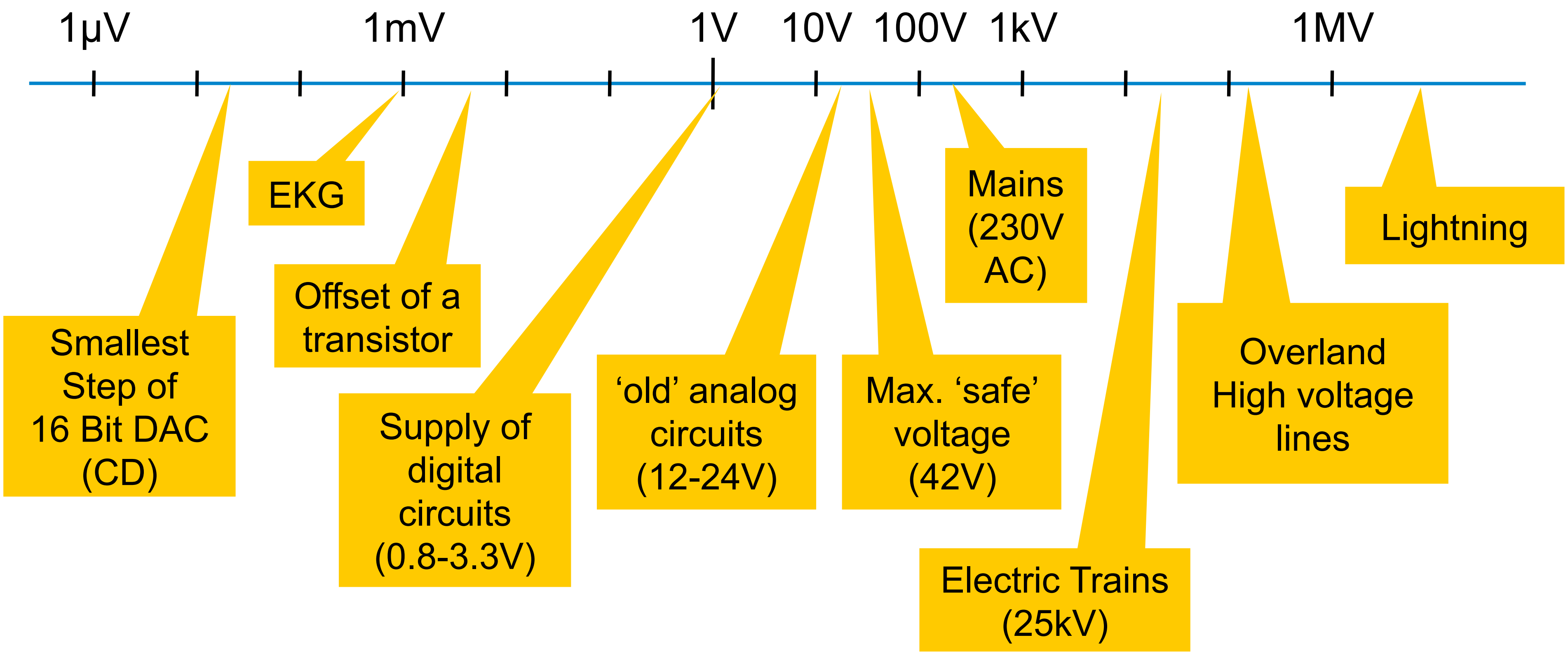
VOLTAGE, CURRENT, KIRCHHOFF'S LAWS

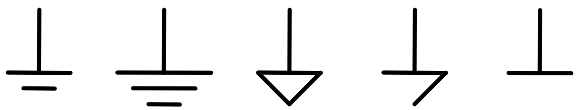
Voltage

- Voltage is the *difference* in electrical potentials, i.e. the energy required to move a unit charge in an electric field
  - This is only well defined in static fields where rot $\vec{E} = \vec{0}$
- Unit: Volt (V)



Ground

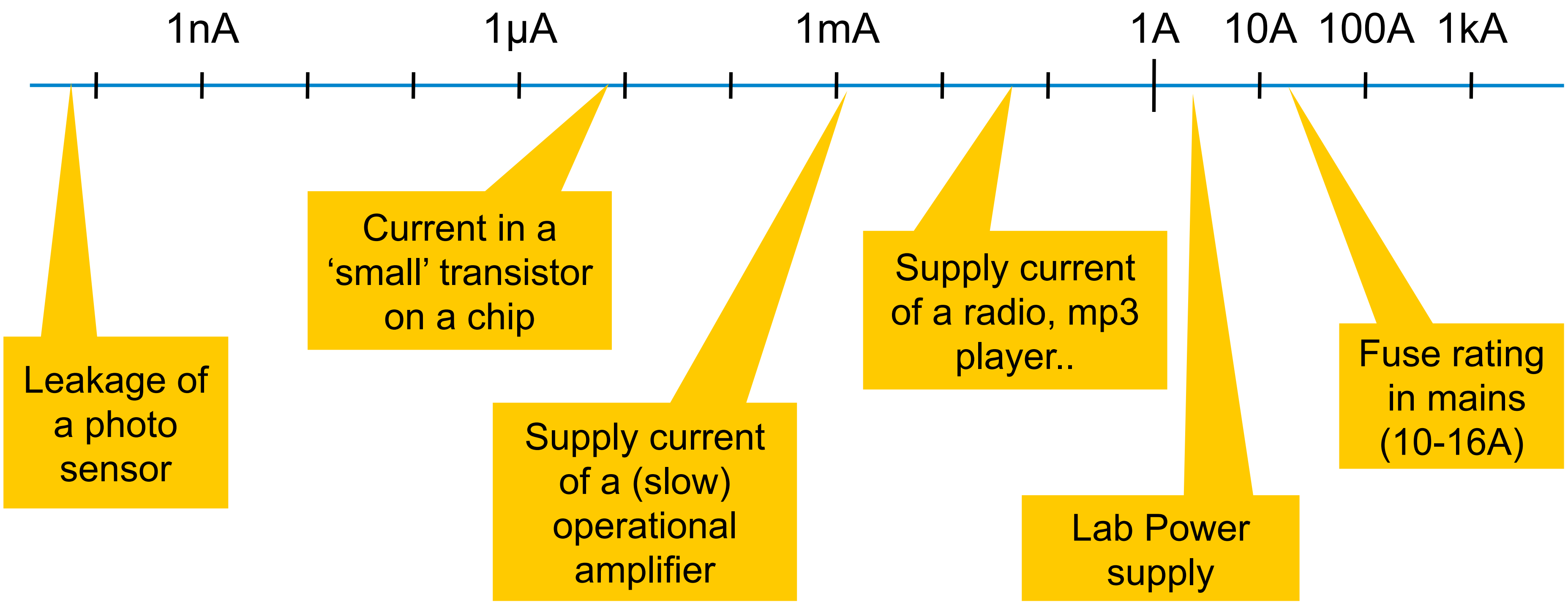
- Voltages are really potential **differences**
- To simplify life, we define a **reference potential** to which voltages are referred. We call it '**ground**'
  - i.e. when we say 'net A has 3V', we mean $V_A - V_{GND} = 3V$
  - Ground is at 0V by definition
- Common ground symbol are:



- (Later we may use several grounds, all at 0V, but separated, for digital and analogue circuit parts)

Current

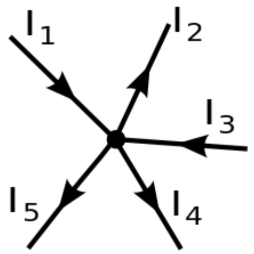
- Electric current is the flow (or change) of electric charge
- i = dQ / dt
- Unit: Ampere (A)



Kirchhoff's Laws

1. The sum of currents at any node is zero:

$$\sum_{k=1}^{n} I_k = 0$$

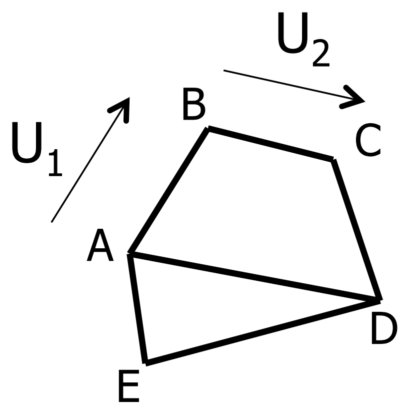


- Follows from charge conservation

2. The sum of voltages in any closed loop is zero:

$$\sum_{k=1}^{n} U_k = 0$$

The sign of the U_k is fixed by a consistent ordering of the nodes in the loop.
Example:
$U_1 = U_B - U_A$, $U_2 = U_C - U_B$, ..
$U_1 + U_2 + U_3 + U_4 = 0$



- Follows from energy conservation

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RESISTORS & CAPACITORS

Resistors

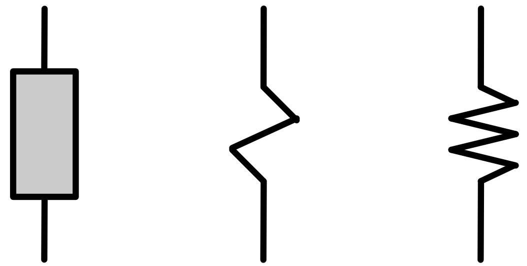
- A resistor is a 2 terminal device
- When voltage is applied, a current flows
- The current is **proportional** to the voltage (Ohms's law):

$I = U \times G$ G is the **conductivity** (Leitwert) in Siemens [S]

or

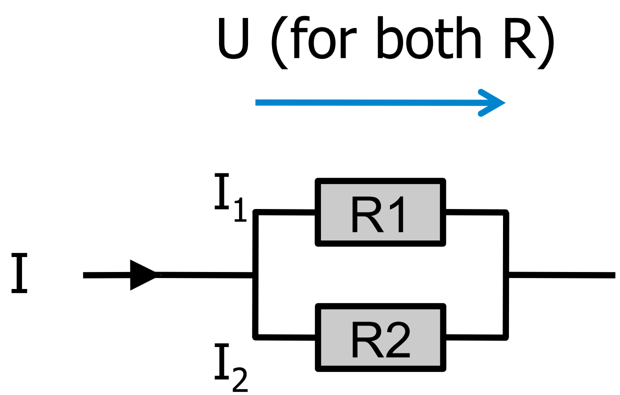
$I = U / R$ R is the **resistivity** (Widerstand) in Ohm [Ω]

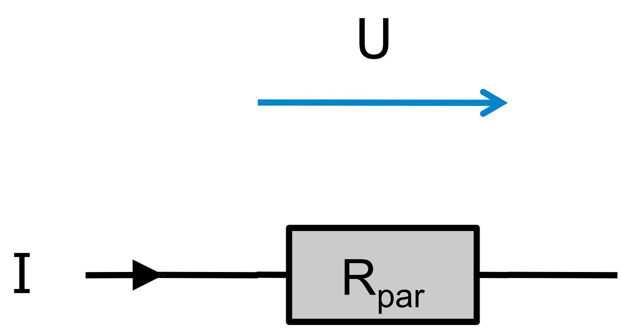
- G and R describe the same relation. G = 1/R, R = 1/G
- Symbols:



- Note: **Ohm's law is not trivial**. Not all materials are 'ohmic'

Parallel Connection of Resistors



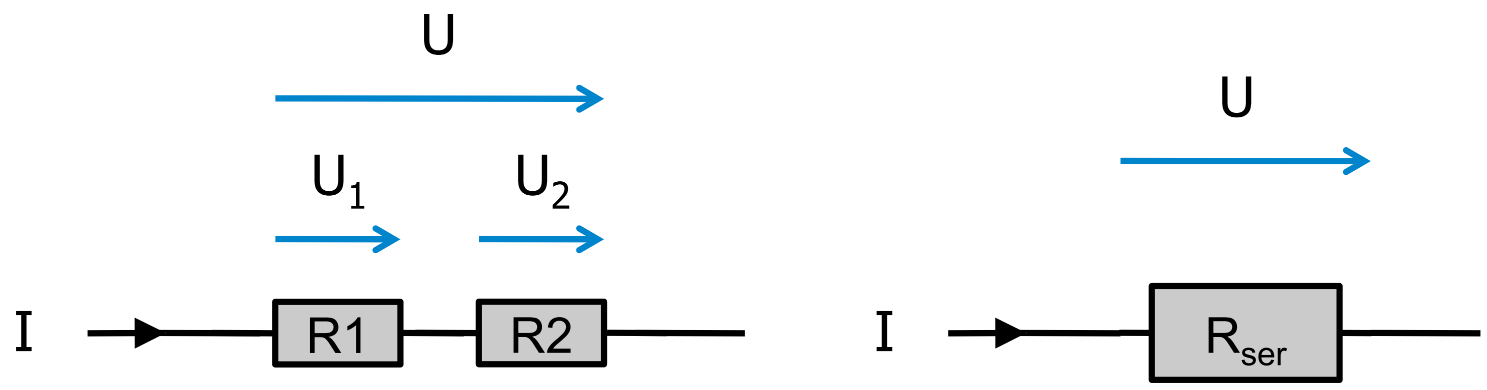


$$I = I_1 + I_2 = G_1 \times U + G_2 \times U = (G_1 + G_2) \times U$$

$$I = G_{par} \times U$$

$$G_{par} = G_1 + G_2 \quad \leftrightarrow \quad 1/R_{par} = 1/R_1 + 1/R_2$$

Series Connection of Resistors

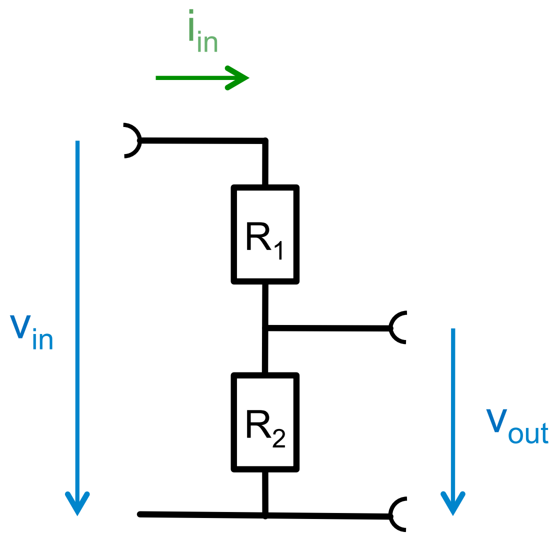


$$U = U_1 + U_2 = I \times R_1 + I \times R_2 = I \times (R_1 + R_2) \qquad U = I \times R_{ser}$$

$$R_{ser} = R_1 + R_2 \quad \leftrightarrow \quad 1/G_{ser} = 1/G_1 + 1/G_2$$

The Voltage Divider (***without*** load current!)

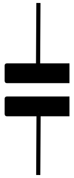
- A very common topology is the voltage divider:

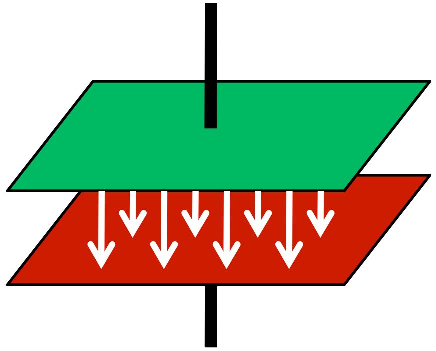


  - The input current $i_{in} = v_{in} / (R_1+R_2)$
  - This current flows through R_1 and R_2, i.e. $i_{in} = i_{R1} = i_{R2}$
  - On R_2, it develops a voltage $v_{out} = i_{R2} R_2 = v_{in} R_2 / (R_1+R_2)$

- Overall: $v_{out} / v_{in} = R_2 / (R_1+R_2)$

Capacitors





- A capacitor can store electrical charge
- Prototype: parallel plate capacitor
  - **Charge Q** on plates generates field (through Gauss' law)
  - Field between plates gives a **voltage V**
- Q = C × V: capacitance is factor between charge and voltage
  - A **large** capacitor can store a **lot** of charge at **low** voltage
- The voltage on a capacitor is given by the current integral:

$$V = \frac{Q}{C} = \frac{1}{C}\int I(t)dt \quad \Leftrightarrow \quad I(t) = C\frac{dV}{dt}$$

- The stored energy is:

$$dE(Q) = V(Q)dQ \Rightarrow E = \int_0^Q V(Q')dQ' = \int_0^Q \frac{Q'}{C}dQ' = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

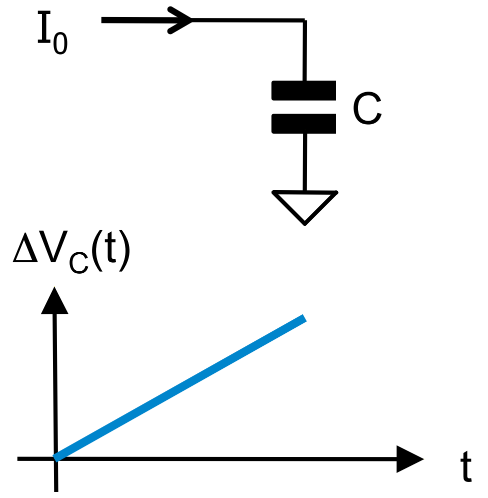
Charging a Capacitor

- At constant current I: **linear ramp**:

$$I(t) = I_0 = \text{const}$$

$$\Delta Q(t) = \int_0^t I(t')\,dt' = \int_0^t I_0\,dt = I_0 \times t$$

$$\Delta U(t) = \frac{\Delta Q(t)}{C} = \frac{I_0}{C} \times t$$

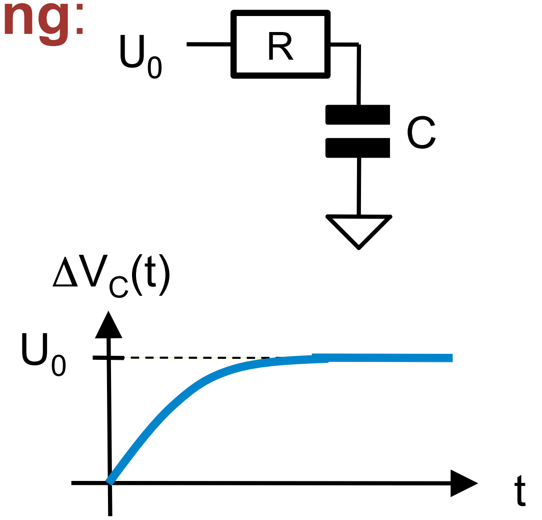


- Through resistor R: **exponential settling**:

$$I(t) = \frac{U_0 - U(t)}{R}$$

$$\frac{dU(t)}{dt} = \frac{I(t)}{C} = \frac{U_0 - U(t)}{RC}$$

$$\text{Solution}: U(t) = U_0 - U_0\, e^{-\frac{t}{RC}}$$



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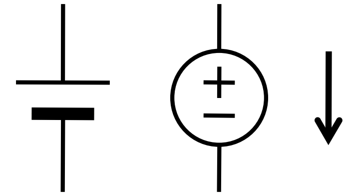
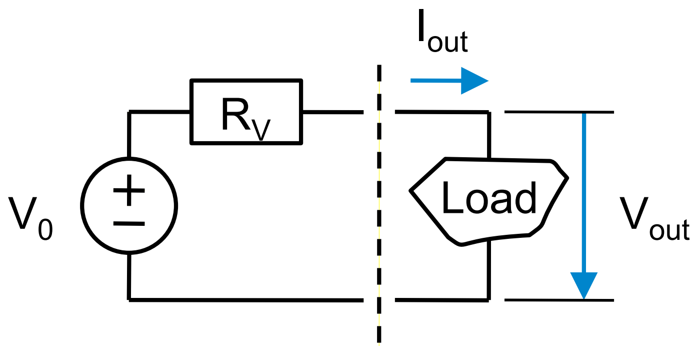
VOLTAGE & CURRENT SOURCES

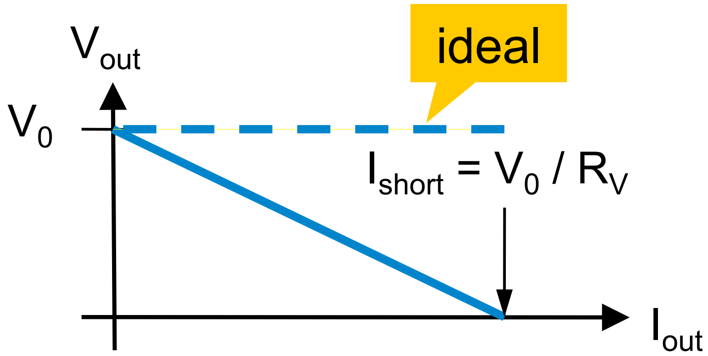
Voltage Sources

- A voltage source has 2 terminals:



- An **ideal** voltage source maintains the voltage for **any** output current ('1000 A')
- The voltage of a **real** source drops with **load current**.
- This is modeled by a **series** resistor (internal resistor, source resistor):

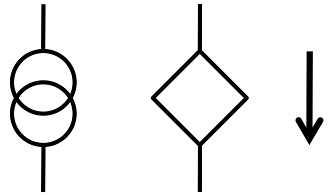




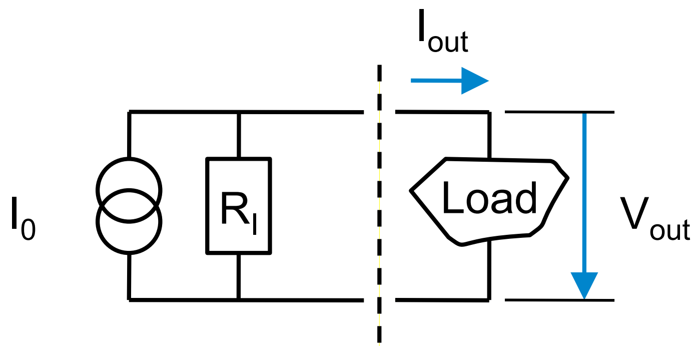
- The **open voltage** is V_0 (I_{out}=0 → voltage drop over R_V is 0)
- The **short circuit current** is $I_{short} = V_0 / R_V$
- Note: '**Good**' voltage sources have **low** $R_V \rightarrow 0$

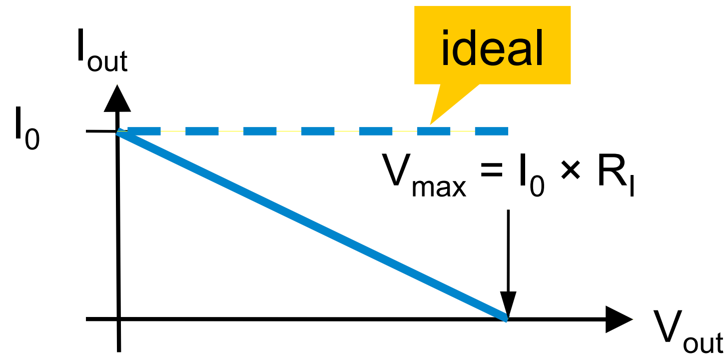
Current Sources

- A current source has 2 terminals:



- An **ideal** current source maintains the current for **any** output voltage
- The current of a **real** source drops with **load voltage**.
- This is modeled by a **parallel** resistor (internal resistor, source resistor):

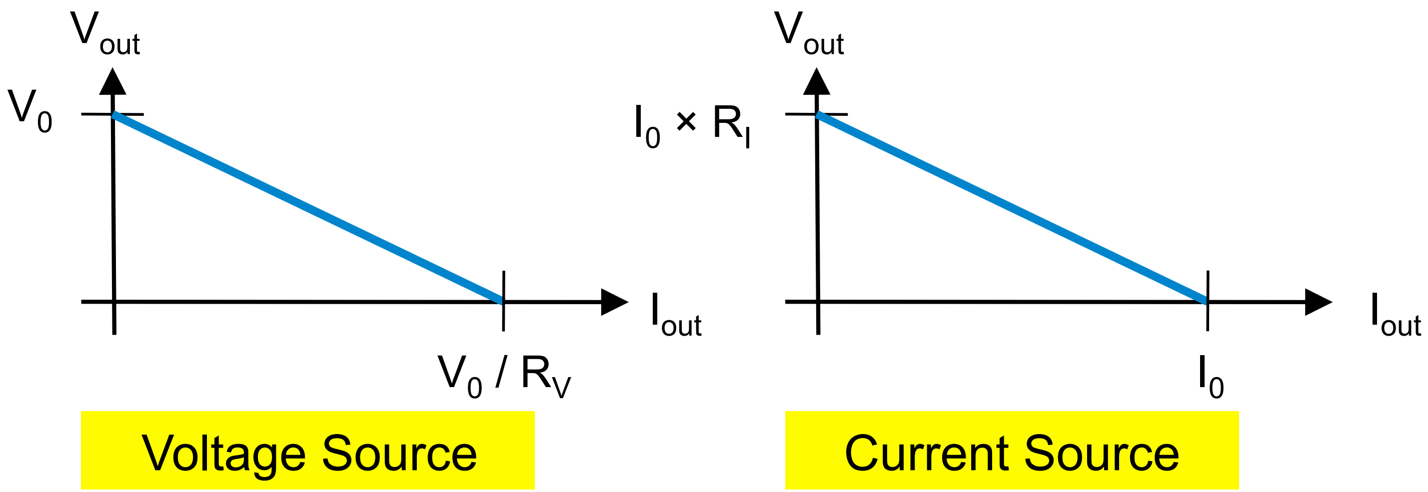




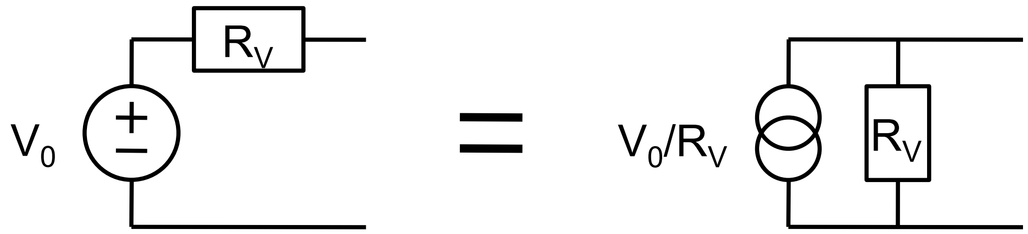
- The **short circuit current** is I_0 (no voltage at R_I→no current)
- At a voltage of $I_0 \times R_I$ no more current flows (all flows in R_I)
- Note: '**Good**' current sources have **high** $R_I \to \infty$

Equivalence of U- and I-Source

- Flip the diagram of the I-source and compare:



- Same shape! Therefore:
- For voltage source with V_0 and R_V, a current source with $I_0 = V_0 / R_V$ and $R_I = V_0 / I_0 = R_V$ behaves the same!

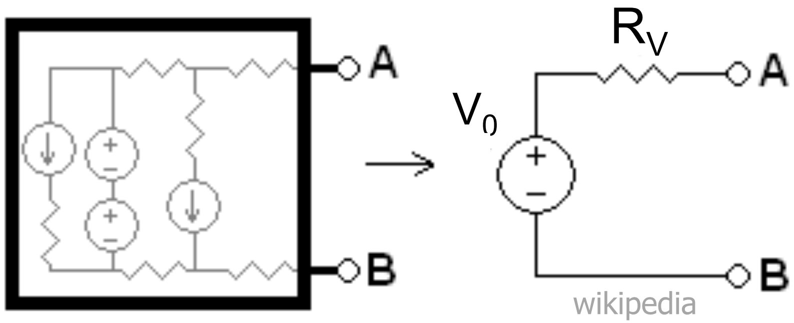


Thévenin’s Theorem

Any combination of U-sources, I-sources and resistors behaves like a (real) voltage source with an internal resistor

- This is fairly obvious from the previous page and the linearity of the resistor properties
- Clearly, a current source with internal resistor can also be used

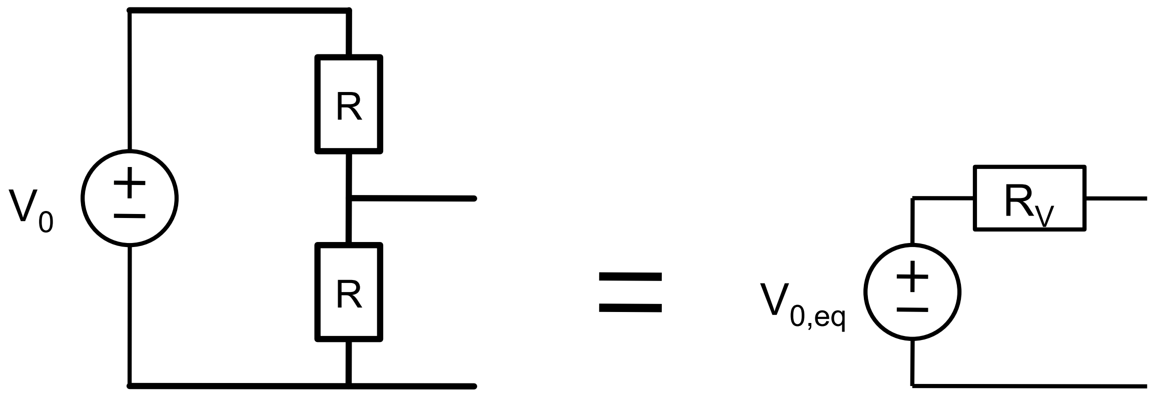
- Example:



- To find V_0: calculate the open voltage
- To find R_V: find the short circuit current. Then $R_V = V_0 / I_{short}$

Thévenin Equivalent of a Voltage Divider

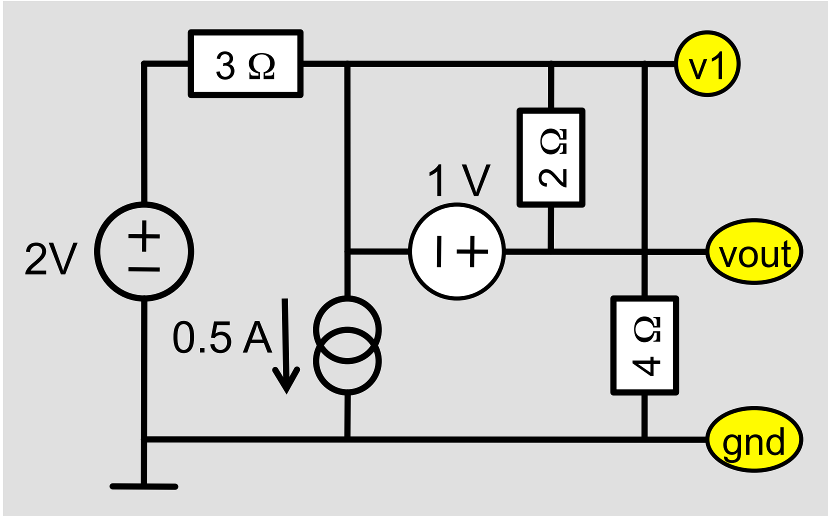
- Consider a voltage divider with equal resistors:



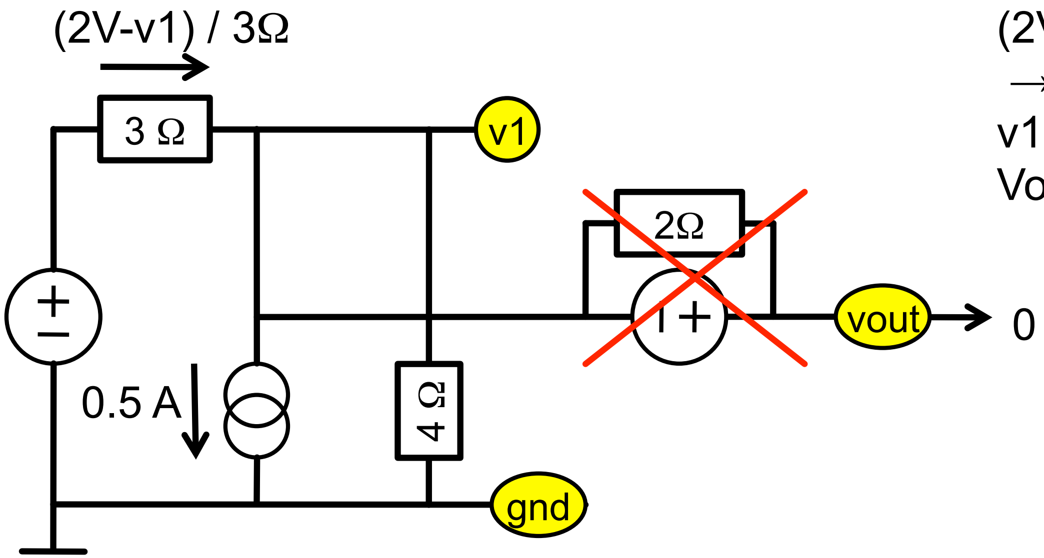
- $V_{0,eq} = V_0 / 2$ $\quad (I = V_0 / (2R),\ V_{0,eq} = R \times I)$
- $I_{short} = V_0 / R \rightarrow R_V = V_{0,eq} / I_{short} = R / 2$

- In general, R_V is the parallel connection of R_1 and R_2

A More Complicated Example - 1



1. Open circuit:



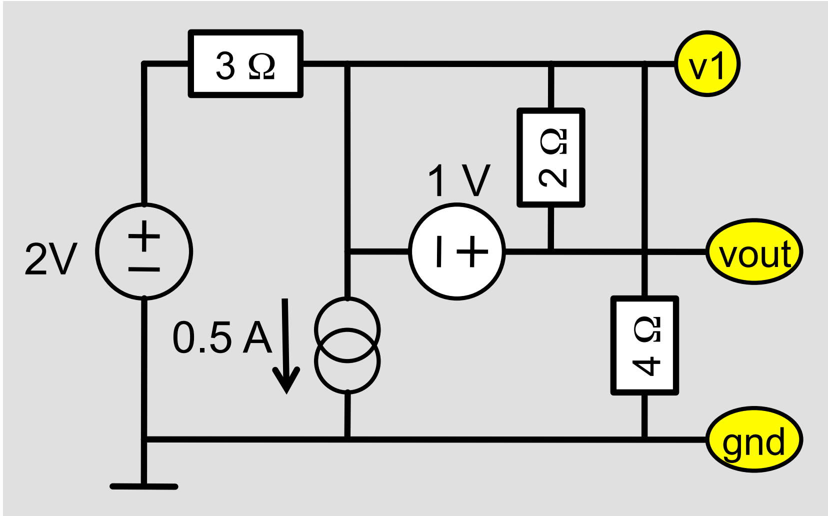
(2V-v1) / 3Ω = 0.5A + v1 / 4Ω

→

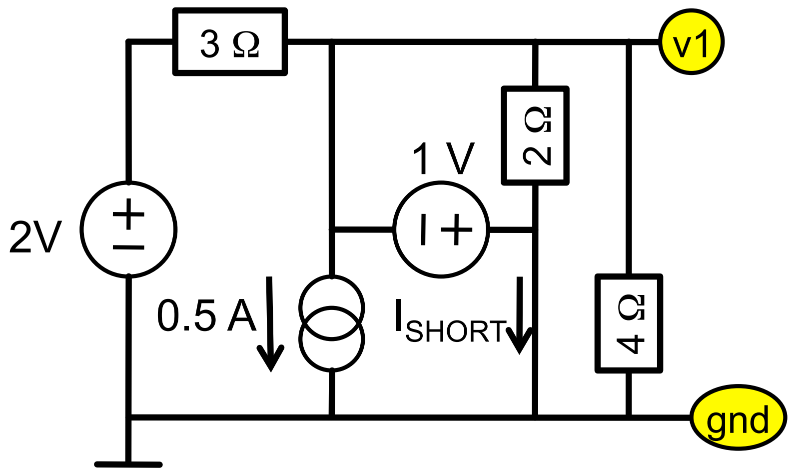
v1 = 0.285714 V

Vout = v1 + 1V = **1.285 V = V_0**

A More Complicated Example - 2



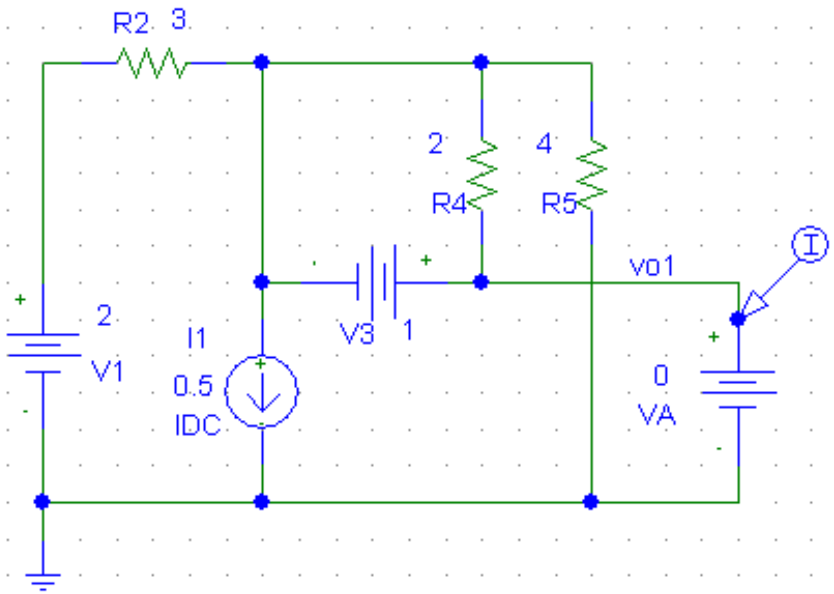
1. Short circuit:

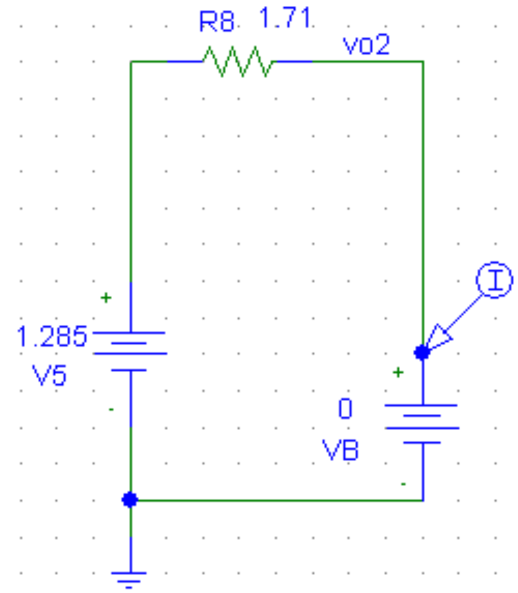


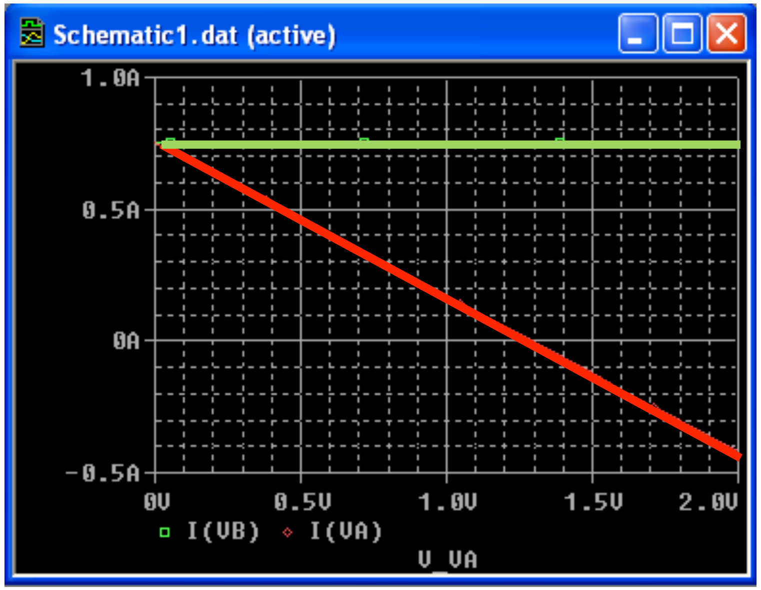
(2V-v1) / 3Ω = 0.5A + v1 / 4Ω + I_{SHORT}
v1 = -1V
→
I_{SHORT}= 0.75 A

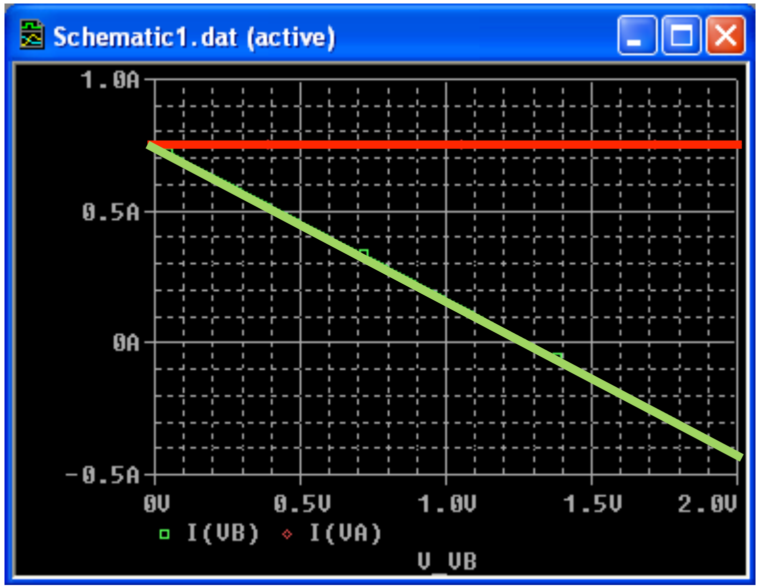
$R_V = V_0 / I_{SHORT}$ = 1.285 V / 0.75 A = **1.71Ω**

A More Complicated Example - Simulation





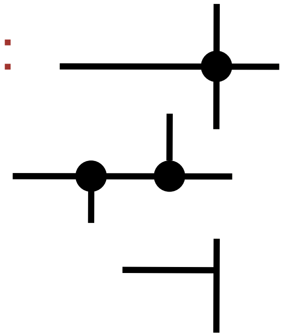


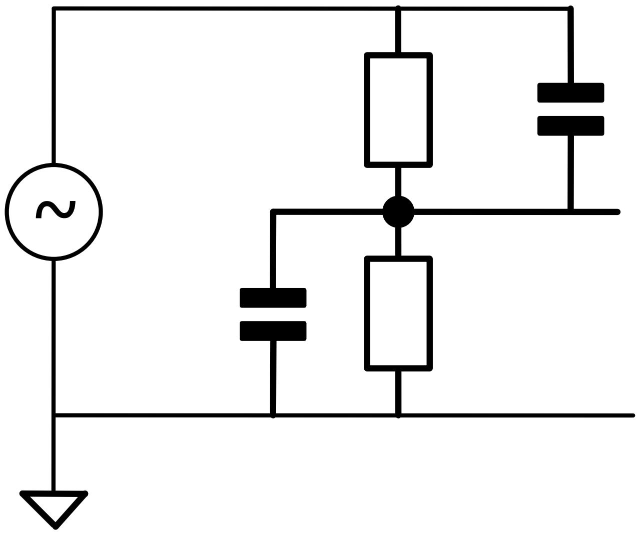


INTERMEZZO: DRAWING SCHEMATICS

Drawing Schematics: Some Rules

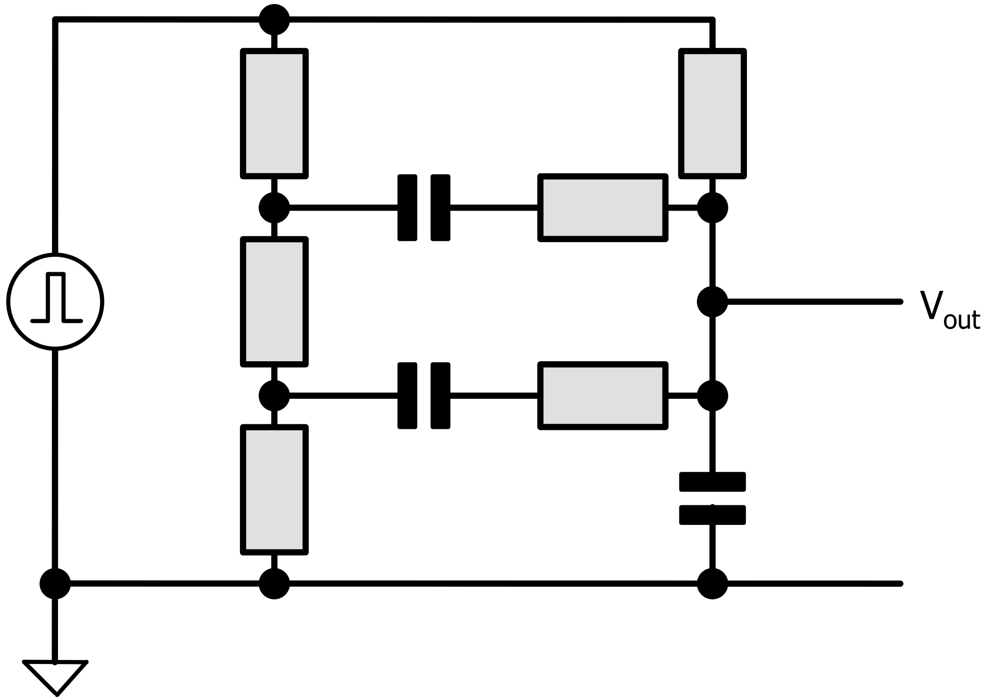
- **Positive** voltages are at the **top**, negative at the bottom
- **Input** signals are at the **left**, **outputs** at the **right**
- Connected **crossings** are marked with a ● :
  - should be avoided
- T-connections do not need a ● :
  - but they can have one…





Example

- A useless circuit…



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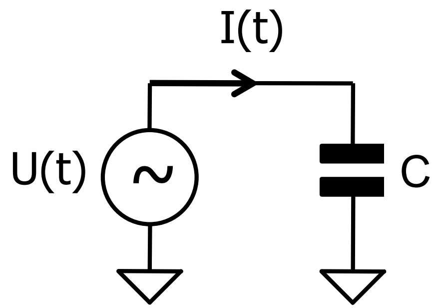
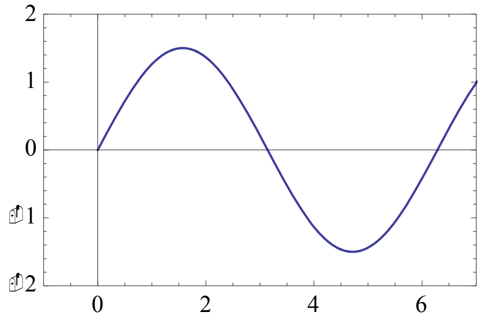
AC BEHAVIOR OF COMPONENTS

AC Behavior of Capacitor

- Consider a capacitor driven by a sine wave voltage:

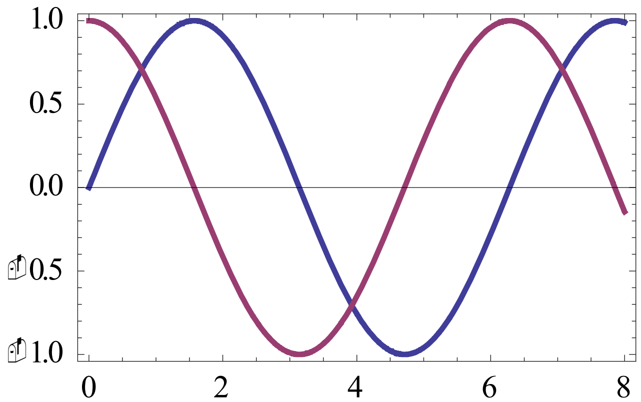
$$U(t) = U_0 \sin(\omega t + \varphi)$$





- The current: $I(t) = C \dfrac{dU(t)}{dt} = C\, U_0\, \omega \cos(\omega t + \varphi)$

is shifted by 90° (sin ↔ cos)!



Complex Impedance

- To simplify our calculations, we would like to extend the relation R= U/I to capacitors, using an **impedance** Z_C .

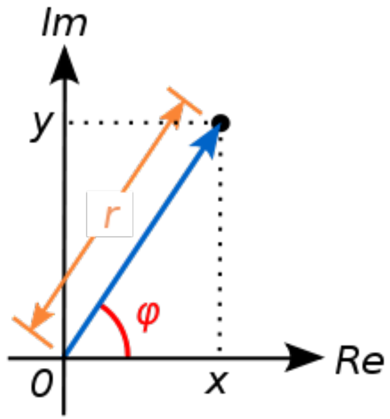
- In order to get the **phase** right, we use **complex** quantities:

$$U(t) = U_0 \sin(\omega t + \varphi) \quad \rightsquigarrow \quad U_0 \cdot e^{i(\omega t + \varphi)} = U_0[\cos(\omega t + \varphi) + i \sin(\omega t + \varphi)]$$

  for voltages and currents.
  - By mixing complex and real parts, we can mix sin() and cos() components and therefore influence the phase.

- Note: Often ‘j’ is used instead of ‘i’ for the complex unit, because ‘i’ is used as current symbol…

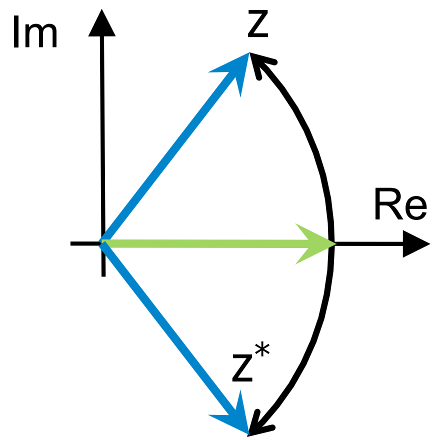
- **Often ‘s’ is used for iω (or jω)**



From Complex Values back to **Real** Quantities

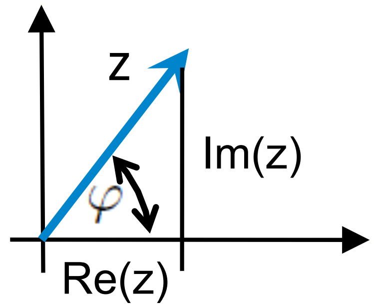
- To find ('back') the **amplitude** of such a complex signal, we calculate the length (**magnitude**) of the complex vector as

$$a = \sqrt{zz^*}$$



- To get the **phase**, we use real and imaginary parts:

$$\varphi = \operatorname{atan}\left(\frac{\mathrm{Im(z)}}{\mathrm{Re(z)}}\right)$$



Note: this simple formula works only in 2 quadrants. You may have to look at signs of Re(z) and Im(z)

Complex Impedance of the Capacitor

- We know that

$$I(t) = C\frac{dU(t)}{dt}$$

- With $U(t) = U_0 \cdot e^{i(\omega t+\varphi)}$

  we have $I(t) = CU'(t) = C \cdot U_0 \cdot i\omega \cdot e^{i(\omega t+\varphi)}$

- Therefore

$$Z_C = \frac{U(t)}{I(t)} = \frac{1}{i\omega C} = \frac{1}{sC}$$

The impedance of a capacitor becomes very small at high frequencies

- Similar: $Z_L = i\,\omega L = s\,L$

Checking this for a Capacitor

- For an input voltage (sine wave of freq. ω) with phase = 0

$$U(t) = U_0\, e^{i\omega t}$$

we have

$$I(t) = \frac{U(t)}{Z_C} = U_0\, e^{i\omega t} \cdot i\omega C$$

- The amplitude of I(t) is

$$\begin{aligned} |I| &= \sqrt{I(t) I^*(t)} \\ &= \sqrt{U_0\, e^{i\omega t} \cdot i\omega C \times U_0\, e^{-i\omega t} \cdot (-i)\omega C} \\ &= \sqrt{U_0^2\, e^{i\omega t} e^{-i\omega t} \cdot (i\omega C)(-i\omega C)} \\ &= U_0 \omega\, C \end{aligned}$$

- The phase is:

$$\varphi = \operatorname{atan}\left(\frac{\omega\, C}{0}\right) = \operatorname{atan}(\infty) = \frac{\pi}{2}$$

  - We have dropped the time variant part and the constant U_0

Simplifying even more

- As we have just seen, the $U(t) = U_0 e^{i\omega t}$ propagates trivially to the output.
- We therefore drop this part and just use ‘1’!

Recipe to Calculate Transfer Functions

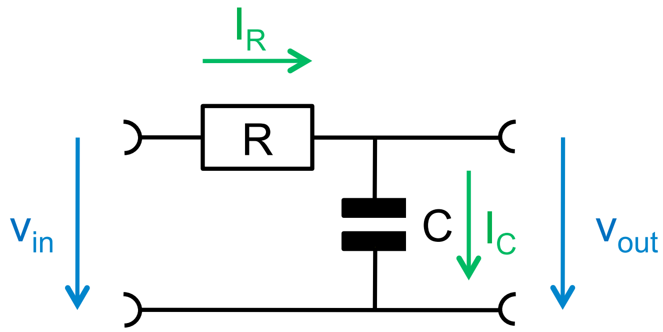
- Replace all component by their complex impedances (1/(sC), sL, R)
- Assume a unit signal of '1' at the input

  (in reality it is $U(t) = U_0 \, e^{i\omega t}$)

- Write down all node current equations or current equalities using Kirchhoff's Law (they depend on s)
  - You need N equations for N unknowns
- Solve for the quantity you are interested in (most often V_{out})
- Analyze the result (amplitude / phase / …)

Example: Low Pass

- Consider



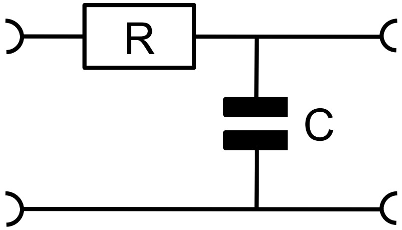
- We have only *one* unknown: v_{out}

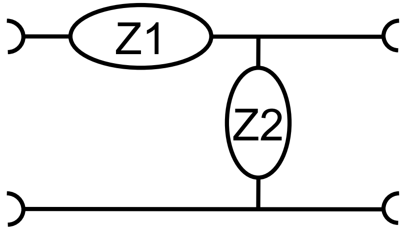
- Current equality at node v_{out}: $$\frac{v_{in} - v_{out}}{R} = I_R = I_C = v_{out}\, s\, C$$

- Solve for v_{out}:

$$
\begin{aligned}
v_{in} - v_{out} &= v_{out}\, s\, C\, R \\
v_{in} &= v_{out}(1 + s\, C\, R) \\
\frac{v_{out}}{v_{in}} &= H(s) = \frac{1}{1 + s\, C\, R}
\end{aligned}
$$

Low Pass as ‘complex’ voltage divider





- This is an ‘ac’ voltage divider with two impedances Z_1 = R and Z_2 = 1/sC

- Using the voltage divider formula, we get

$$H(s) = \frac{\mathrm{v_{out}}}{\mathrm{v_{in}}} = \frac{\mathrm{Z}_2}{\mathrm{Z}_2 + \mathrm{Z}_1} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{1}{1 + s\,RC} = \frac{1}{1 + i\frac{\omega}{\omega_0}}$$

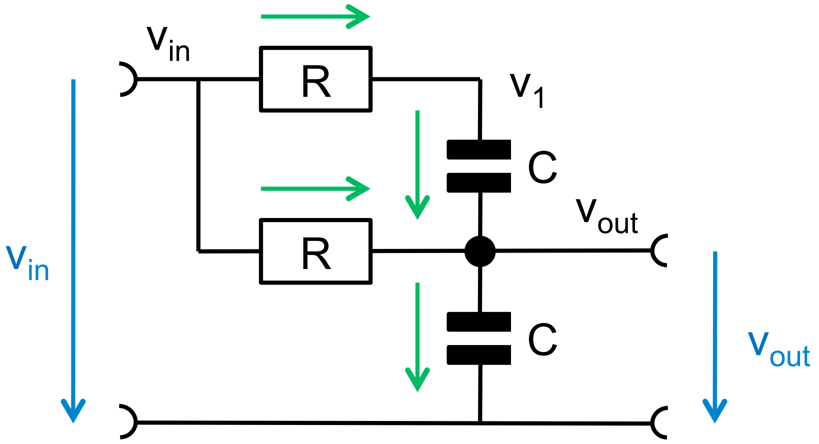
with ω_0 = 1/(RC), the ‘corner frequency’.

The HIGH Pass

- By exchanging R and C, low frequencies are blocked and high frequencies pass through. This is the High-Pass.

- We get $H_{HP}(s) = \dfrac{R}{R + \dfrac{1}{sC}} = \dfrac{sRC}{1 + sRC}$

More Complicated Example



- We have now *two* unknowns: v_1, v_{out}

$$EQ1\,(@v_1) \quad : \quad \frac{v_{in} - v_1}{R} = (v_1 - v_{out})sC$$

$$EQ2\,(@v_{out}) \quad : \quad (v_1 - v_{out})sC + \frac{v_{in} - v_{out}}{R} = v_{out}\, sC$$

- Eliminating v_1 gives:

$$H(s) = \frac{1 + 2RC\,s}{1 + 3RC\,s + (RC)^2\,s^2}$$

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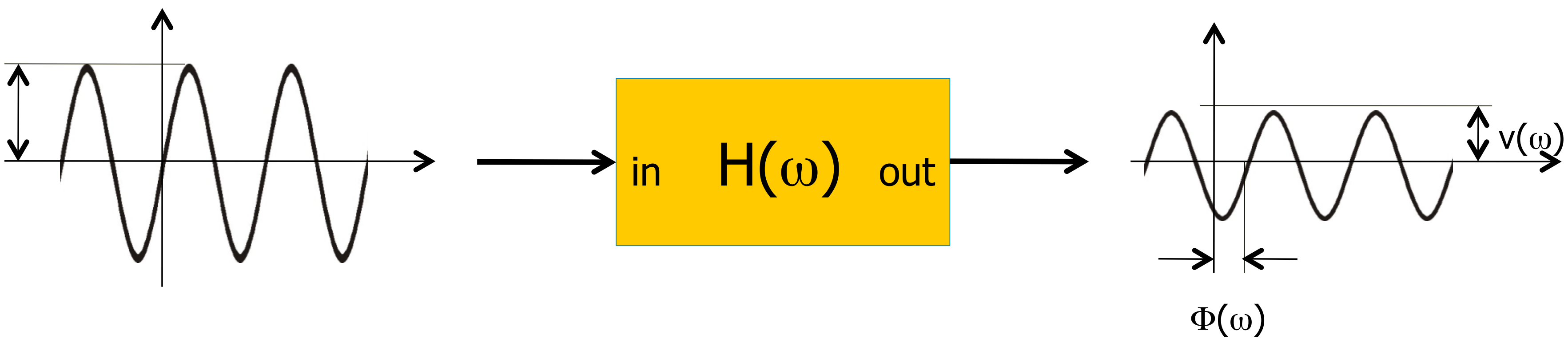
Bode Plot

Transfer Function

- The **transfer function** of a *linear, time invariant* system visualizes how the **amplitude** and **phase** of a sine wave input signal of **constant frequency** ω appears at the output

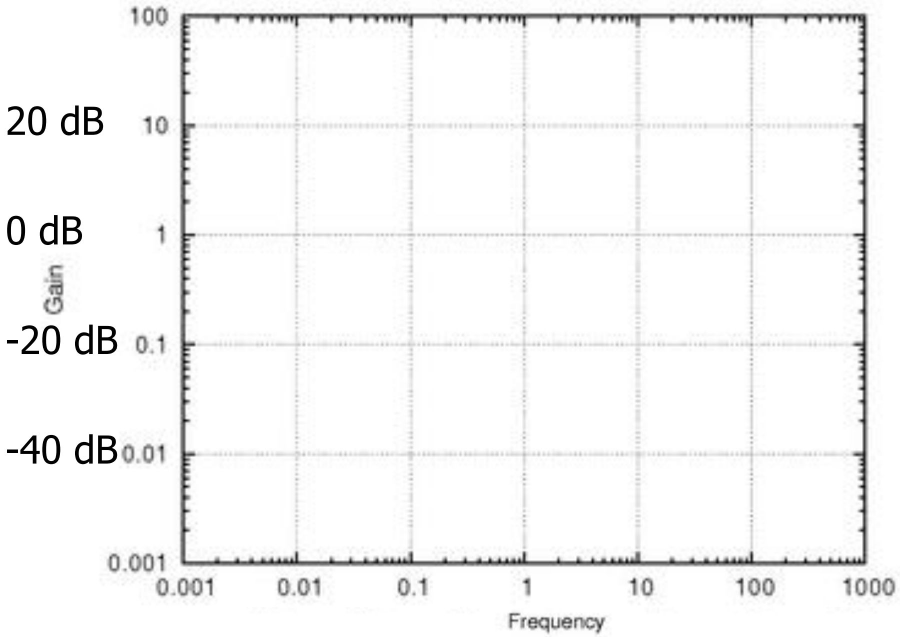
- The frequency remains unchanged
- The transfer function H(ω) contains
  - The phase change Φ(ω)
  - The gain v(ω) = amp_in / amp_out (ω)

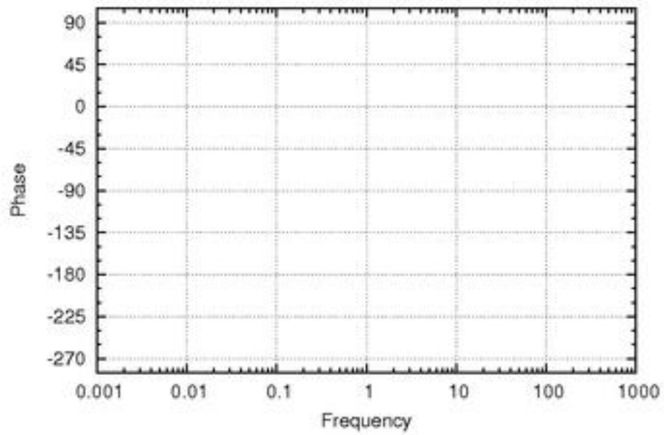


Bode Diagram: Definition

- The Bode Plot shows gain (+ phase) of the transfer function
- The frequency (x-axis) is plotted **logarithmically**
- Gain is plotted (y-axis) **logarithmically**, often in **decibel**
  - $DB(x) = 20 \log_{10} (x)$:

× 10	+20 dB	
× 100	+40 dB	
× 2	6 dB	(not exactly!)
× 1	0 dB	
/ 2	-6 dB	
/ √2	-3 dB	





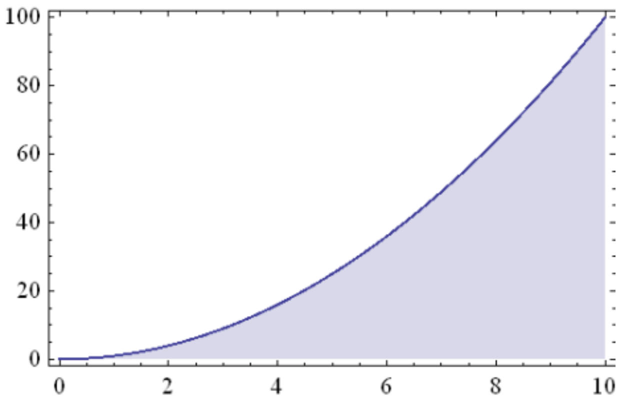
  - dBs for multiplied quantities just add !

Bode Diagram: Properties

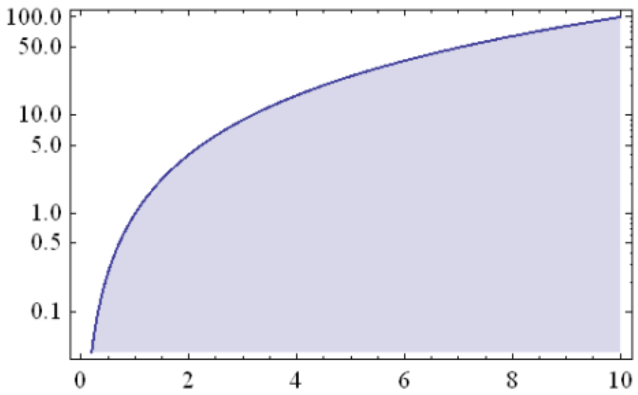
- Power functions are straight lines:

$$f(x) = x^n \Rightarrow \log[f(x)] = n\log(x)$$

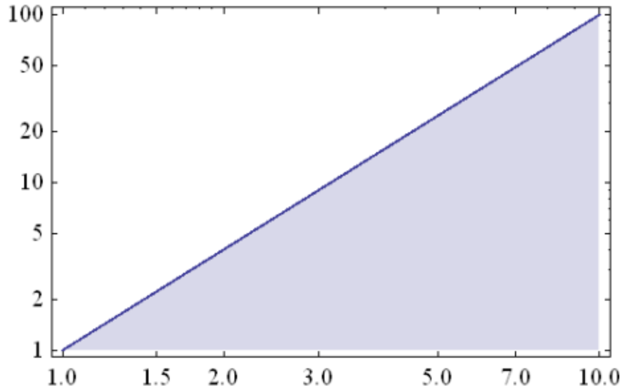
Plot[x^2,{x,0,10}]



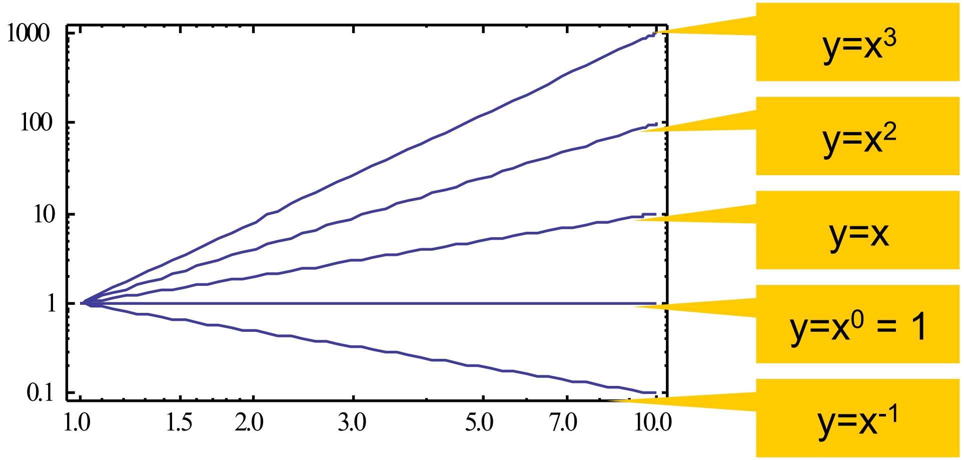
LogPlot[x^2,{x,0,10}]



LogLogPlot[x^2,{x,1,10}]



LogLogPlot[Table[x^N,{N,-1,3}],{x,1,10}]



Bode Diagram: Properties

- 1/x function has slope -1:

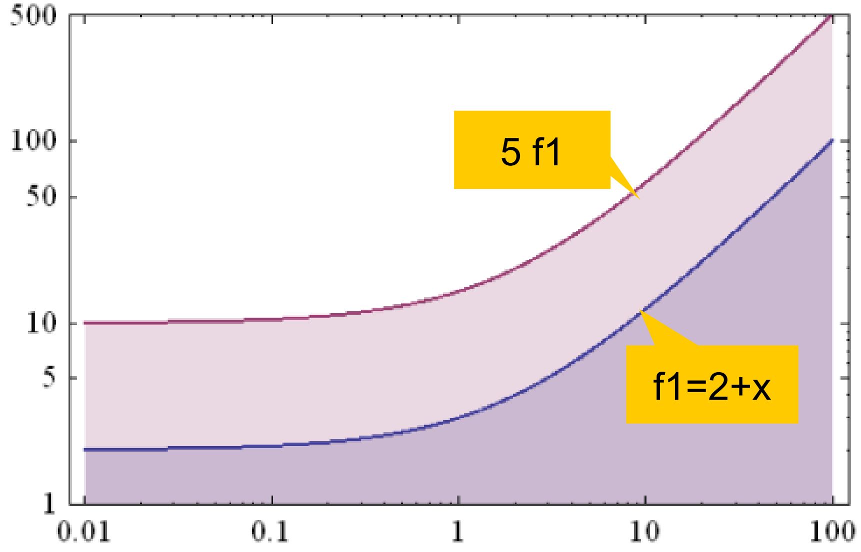
$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow \log[f(x)] = -1 \log(x)$$

- Multiplied functions are **added** in plot:

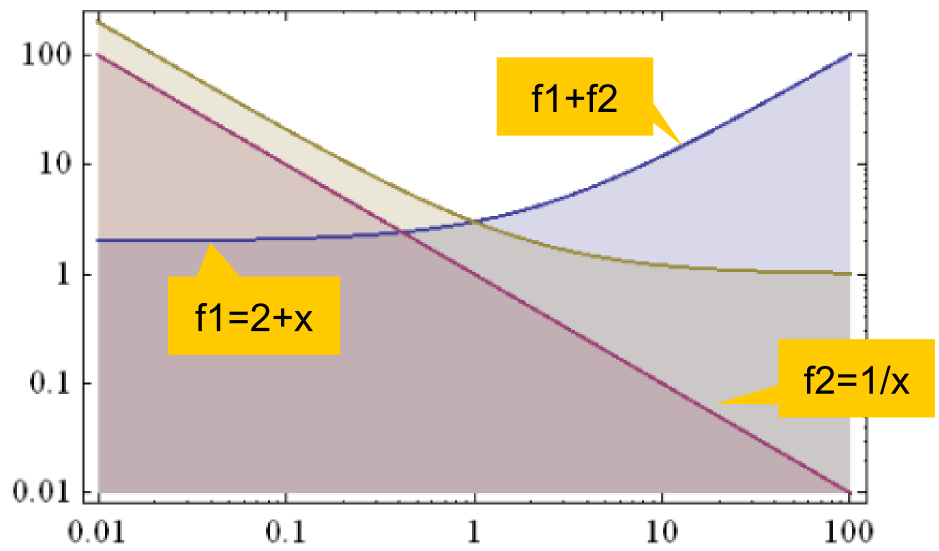
$$f = f_1 \cdot f \Rightarrow \log[f] = \log(f_1) + \log(f_2)$$

f1=2+x;f2=x^{-1};

LogLogPlot[{f1, 5 f1},{x,0.01,100}]



LogLogPlot[{f1, f2,f1 * f2},{x,0.01,100}]



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The Low Pass Filter

Analysis of the Low Pass Transfer Function

- Transfer Function: $$H(\omega) = \frac{1}{1 + i\frac{\omega}{\omega_0}}$$

- Magnitude: $$v(\omega) = \sqrt{H(\omega)H^*(\omega)} = \frac{1}{\sqrt{(1 + i\frac{\omega}{\omega_0})(1 - i\frac{\omega}{\omega_0})}}$$

$$v(\omega) = \frac{1}{\sqrt{(1 + \frac{\omega^2}{\omega_0^2})}}$$

$$\rightarrow 1 \quad \text{for} \quad \omega \rightarrow 0$$
$$\rightarrow \frac{1}{\sqrt{2}} \quad \text{for} \quad \omega = \omega_0$$
$$\rightarrow \frac{\omega_0}{\omega} \quad \text{for} \quad \omega \rightarrow \infty$$

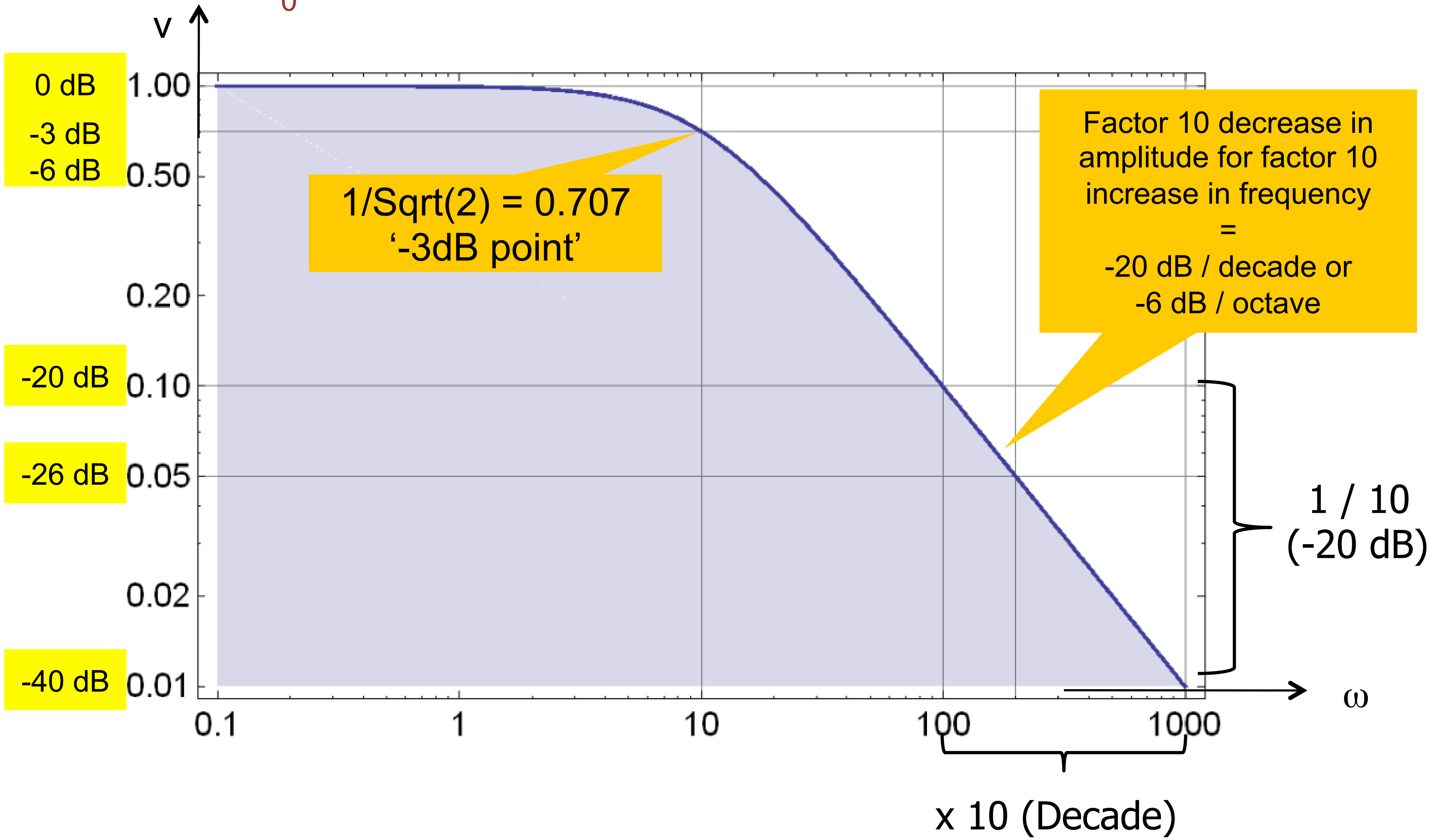
- Phase: $$H(\omega) = \frac{1}{1 + i\frac{\omega}{\omega_0}} = \frac{1}{1 + i\frac{\omega}{\omega_0}} \times \frac{1 - i\frac{\omega}{\omega_0}}{1 - i\frac{\omega}{\omega_0}} = \frac{1 - i\frac{\omega}{\omega_0}}{1 + \frac{\omega^2}{\omega_0^2}}$$

$$\varphi = \mathrm{atan}\left(\frac{\mathrm{Im}(H)}{\mathrm{Re}(H)}\right) = -\mathrm{atan}\left(\frac{\omega}{\omega_0}\right)$$

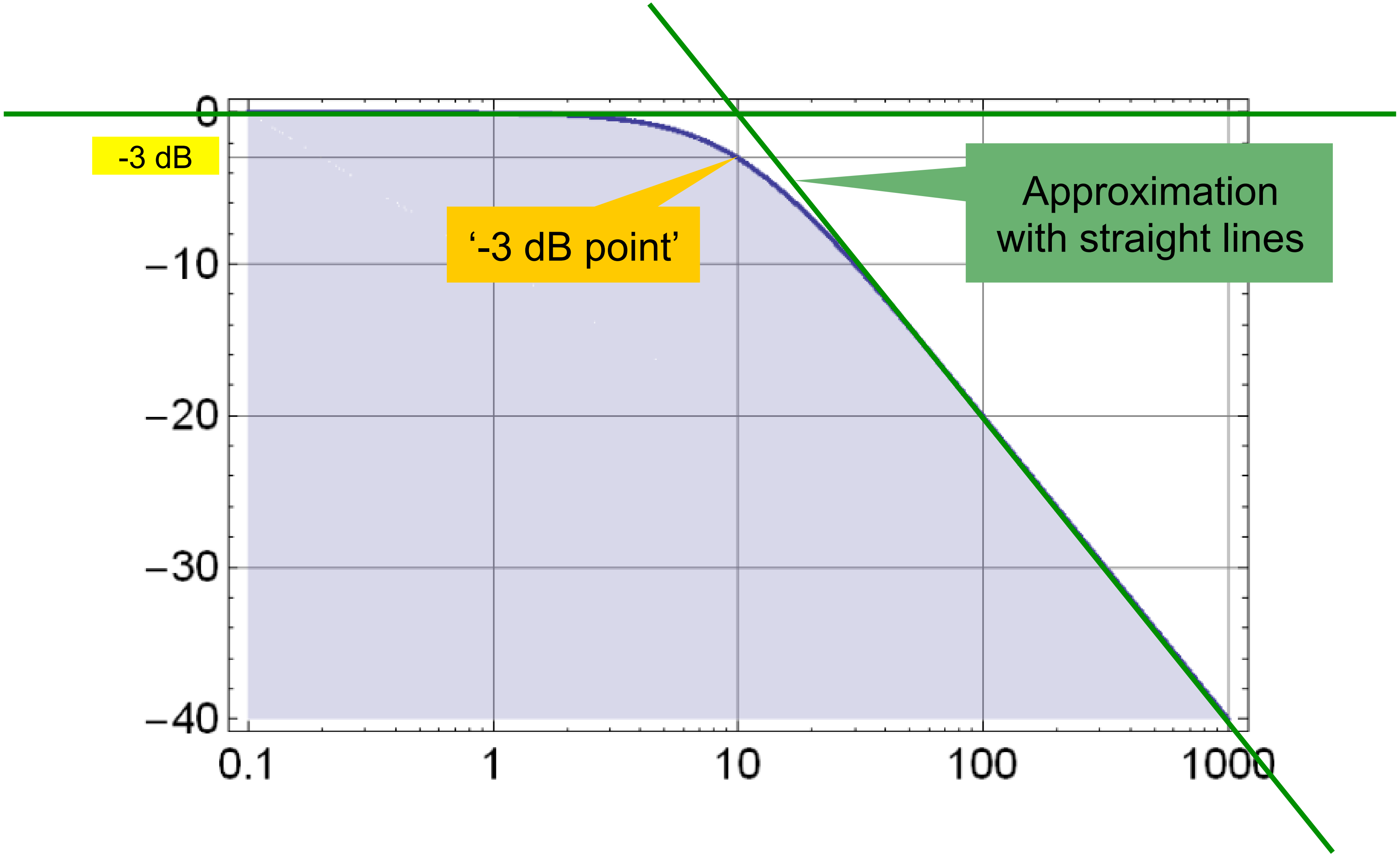
(rad or degree)

Bode Plot of LowPass (Amplitude)

- $\omega_0 = 10$

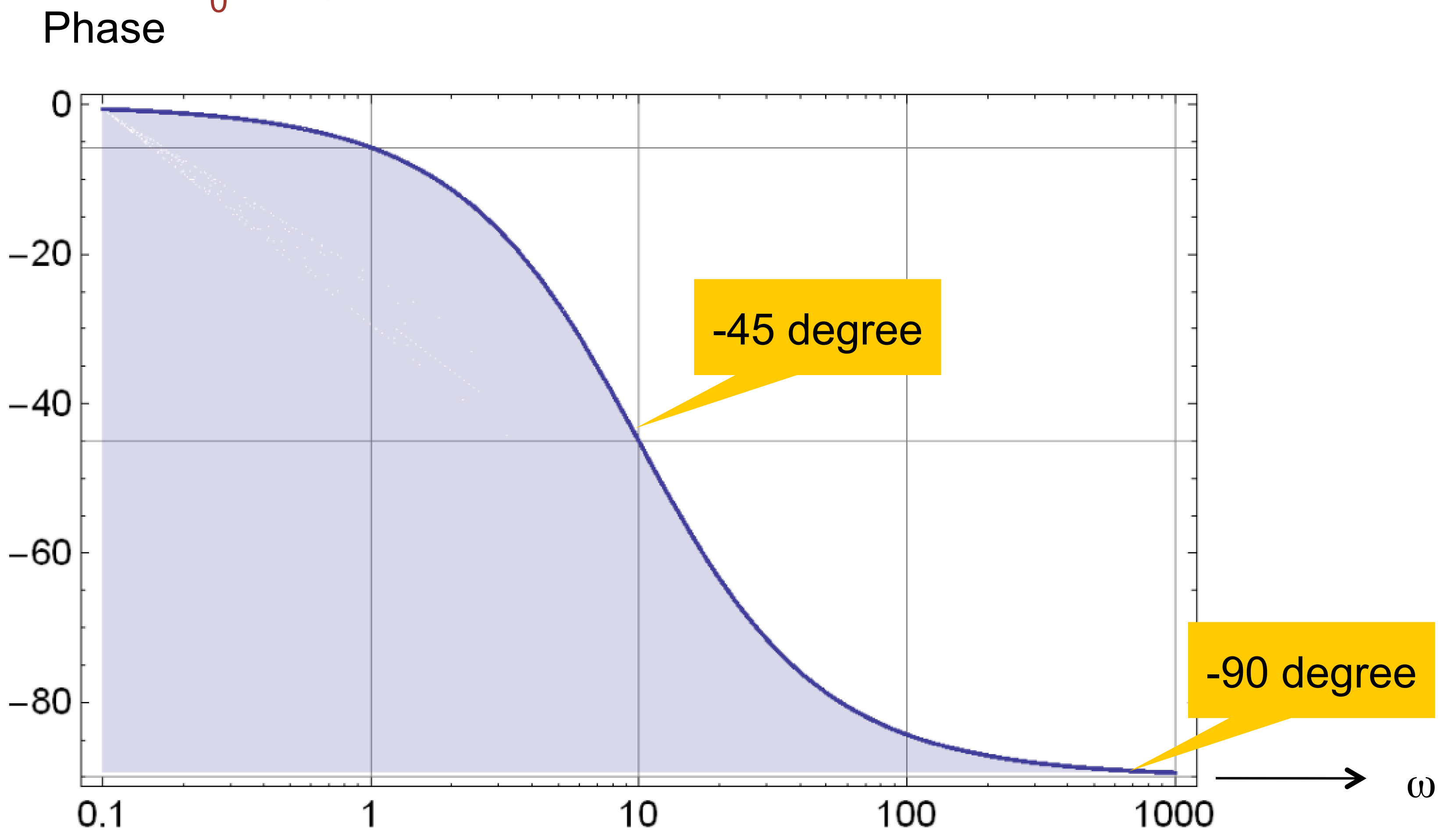


The same in dB



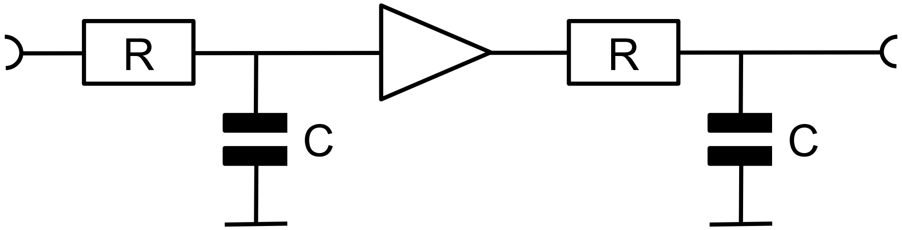
Bode Plot of LowPass (Phase)

- $\omega_0 = 10$

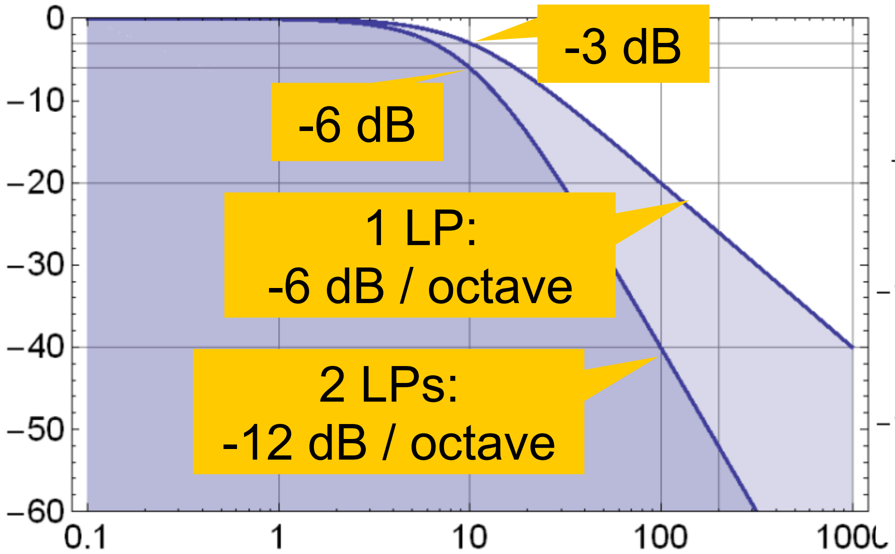


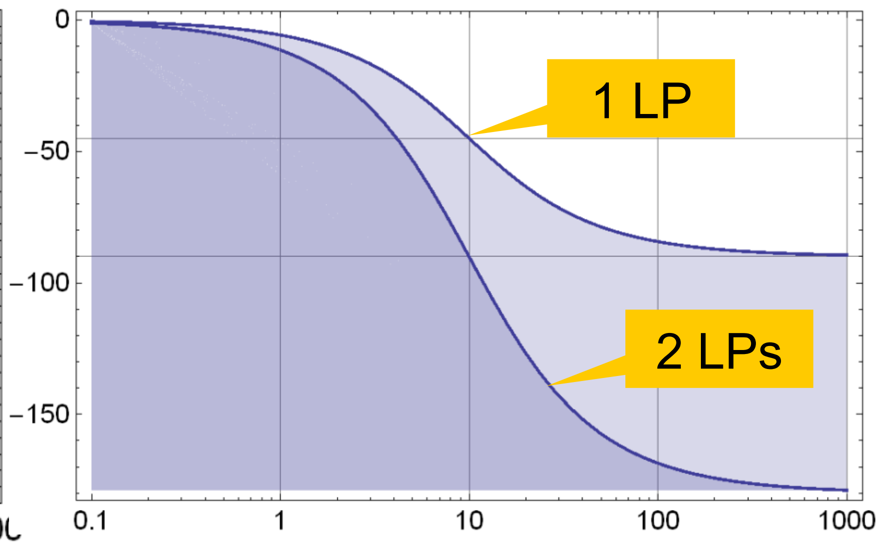
Series Connection of two Low Pass Filters

- Consider two identical LP filters. A 'unit gain buffer' makes sure that the second LP does not load the first one:



- From the properties of the LogLog Plot, the TF of the 2nd order LP is just the sum of two 1st order LPs:



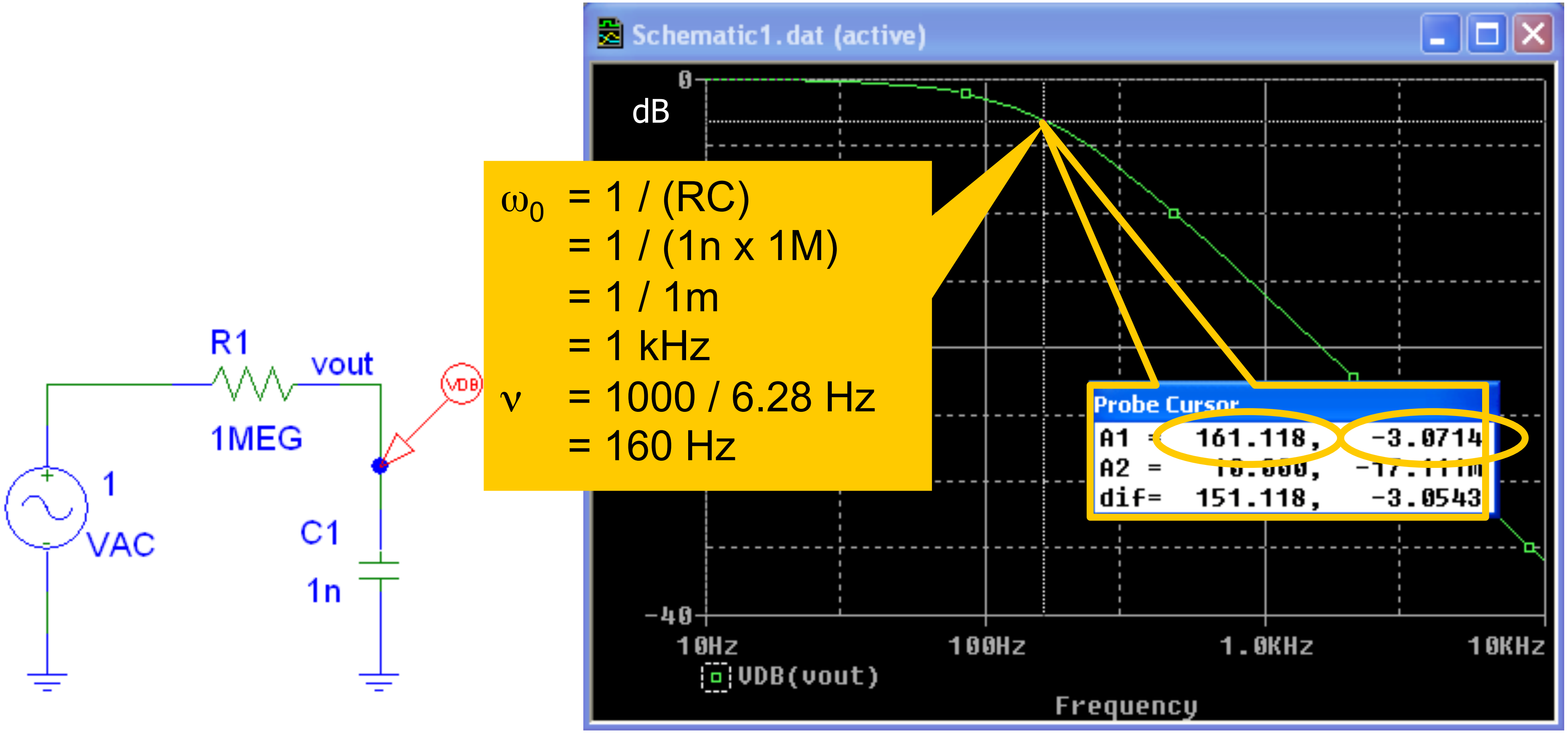


Why bother so much about the low pass ?

- All circuits behave like low-passes (at some frequency)!

Caveat!

- So far, frequency is expressed with ω, i.e. in radian / second
- We have: $\omega = 2 \pi \nu$
- Therefore, the frequencies in Hertz are 2π lower!!!



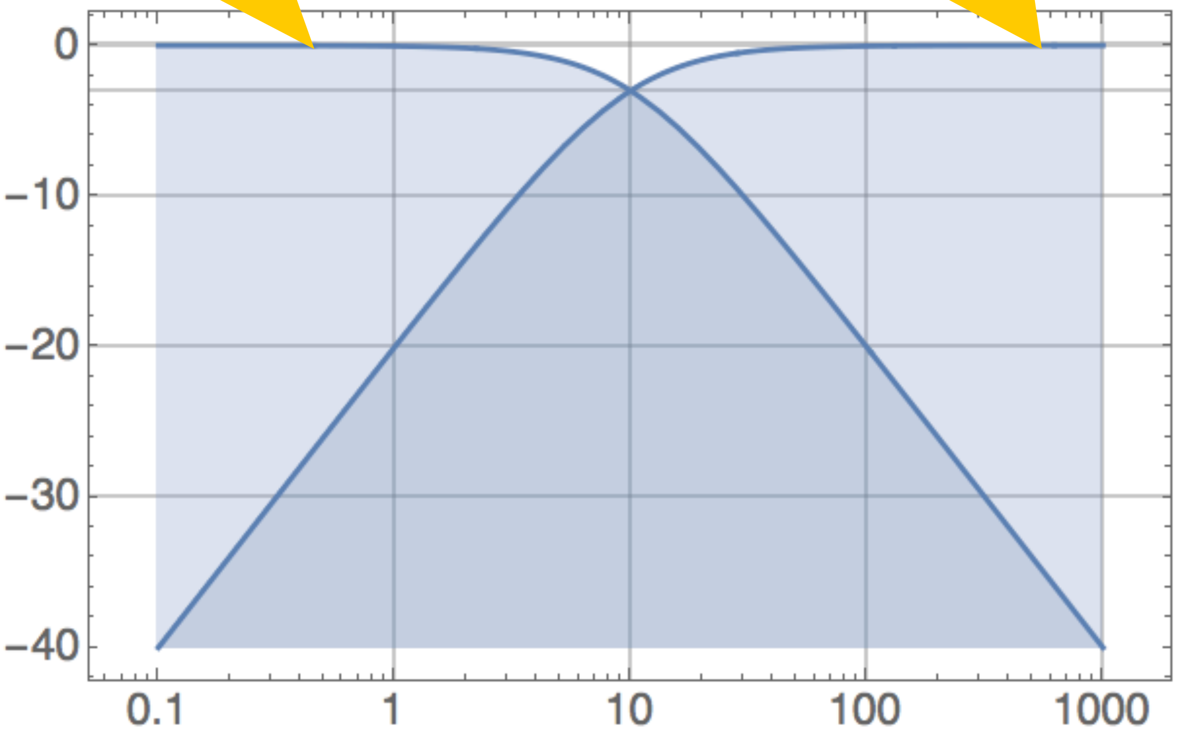
Low Pass and High Pass

$$\mathbf{LP}[\omega] = \frac{1}{1 + i\,\frac{\omega}{\omega 0}}$$

$$\mathbf{HP}[\omega] = \frac{i\,\frac{\omega}{\omega 0}}{1 + i\,\frac{\omega}{\omega 0}};$$

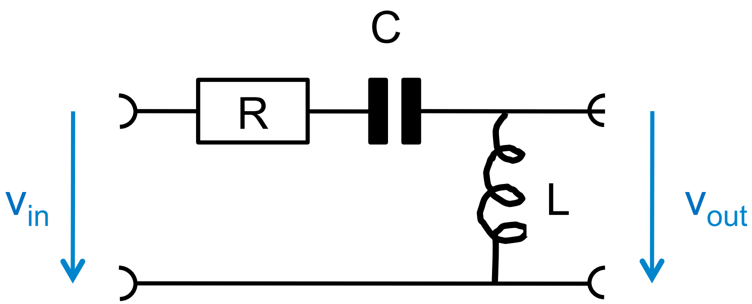
$$\mathbf{LPgain}(\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega 0}\right)^2}}$$

$$\mathbf{HPgain}(\omega) = \frac{\frac{\omega}{\omega 0}}{\sqrt{1 + \left(\frac{\omega}{\omega 0}\right)^2}}$$



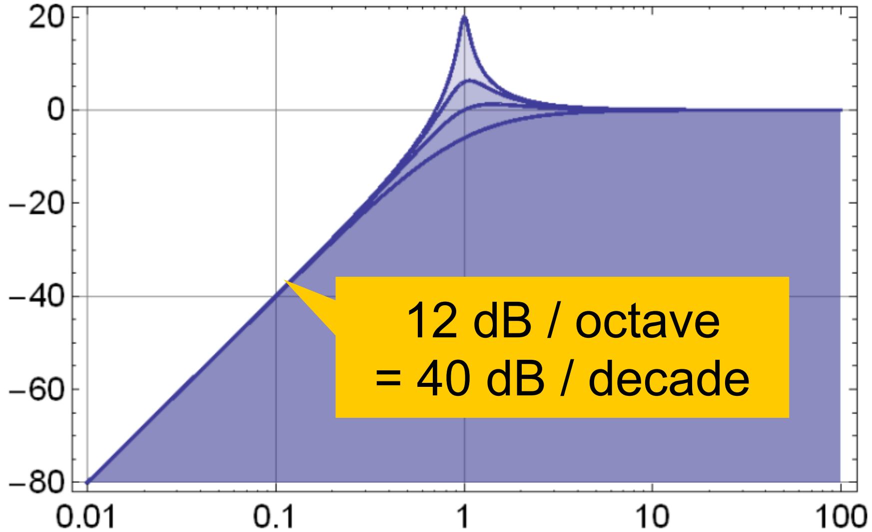
A More Complex Example

- Consider a (High Pass) filter with an inductor:



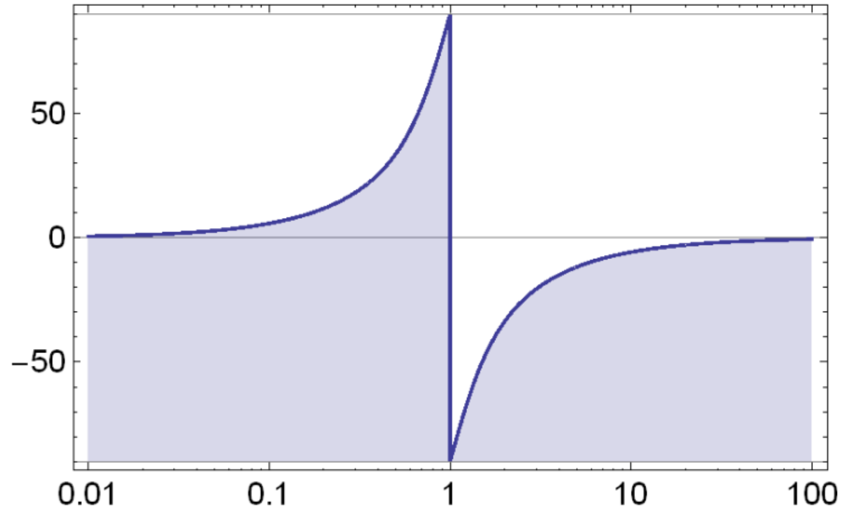
Mathematica Demo

- The transfer function is $H(s) = (C\,L\,s^2)/(1+C\,R\,s+C\,L\,s^2)$
- It is of 'second order' (s has exponent of 2 in denominator)
- Magnitude:
  L=C=1
  R=0.1,0.5,1,2
- 'Inductive peaking'

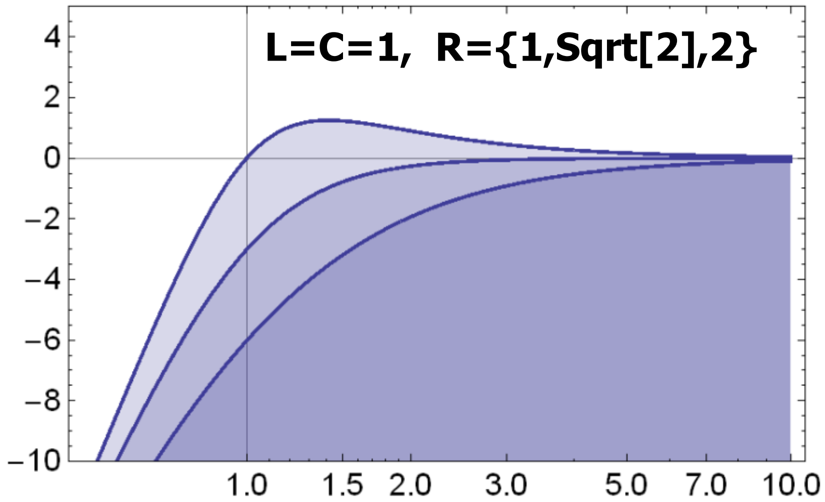


Phase

- Phase has a 'jump'



- For fun:
  - When is filter steep & flat?



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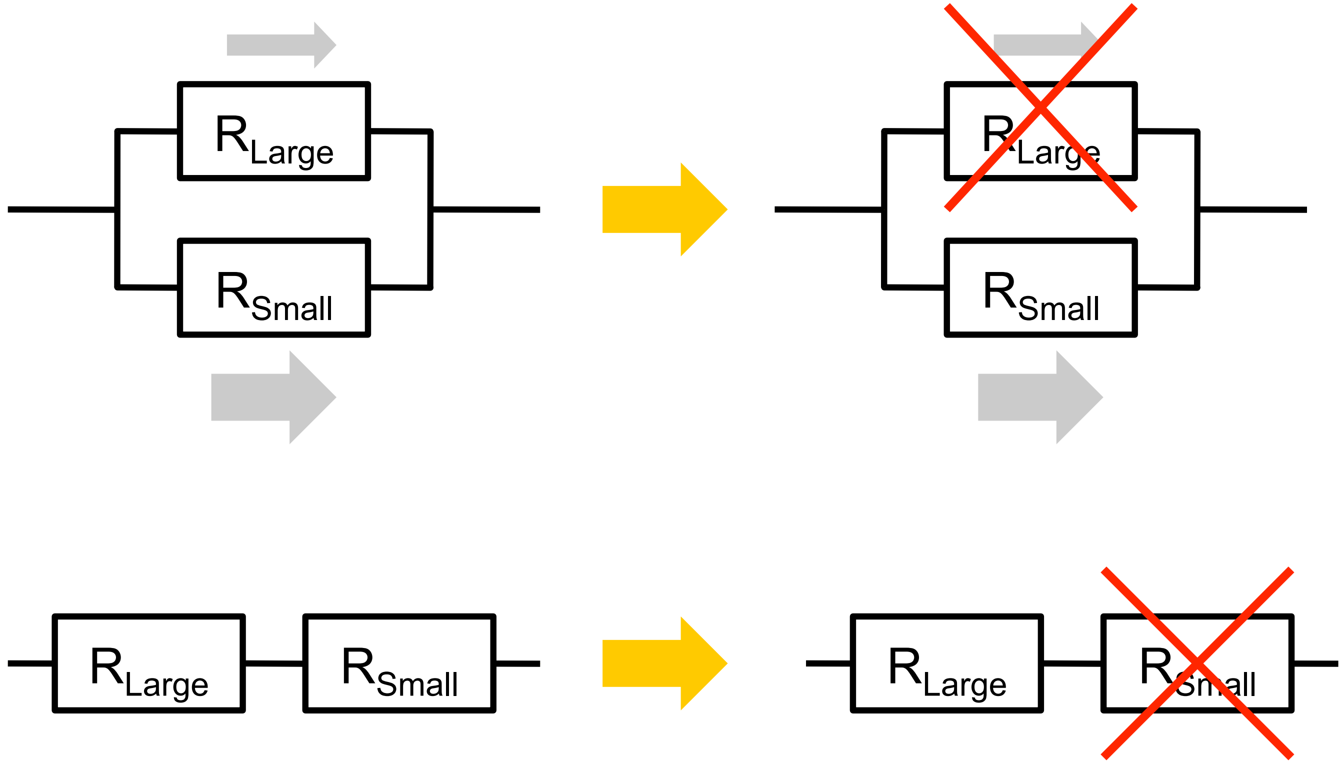
CIRCUIT SIMPLIFICATIONS

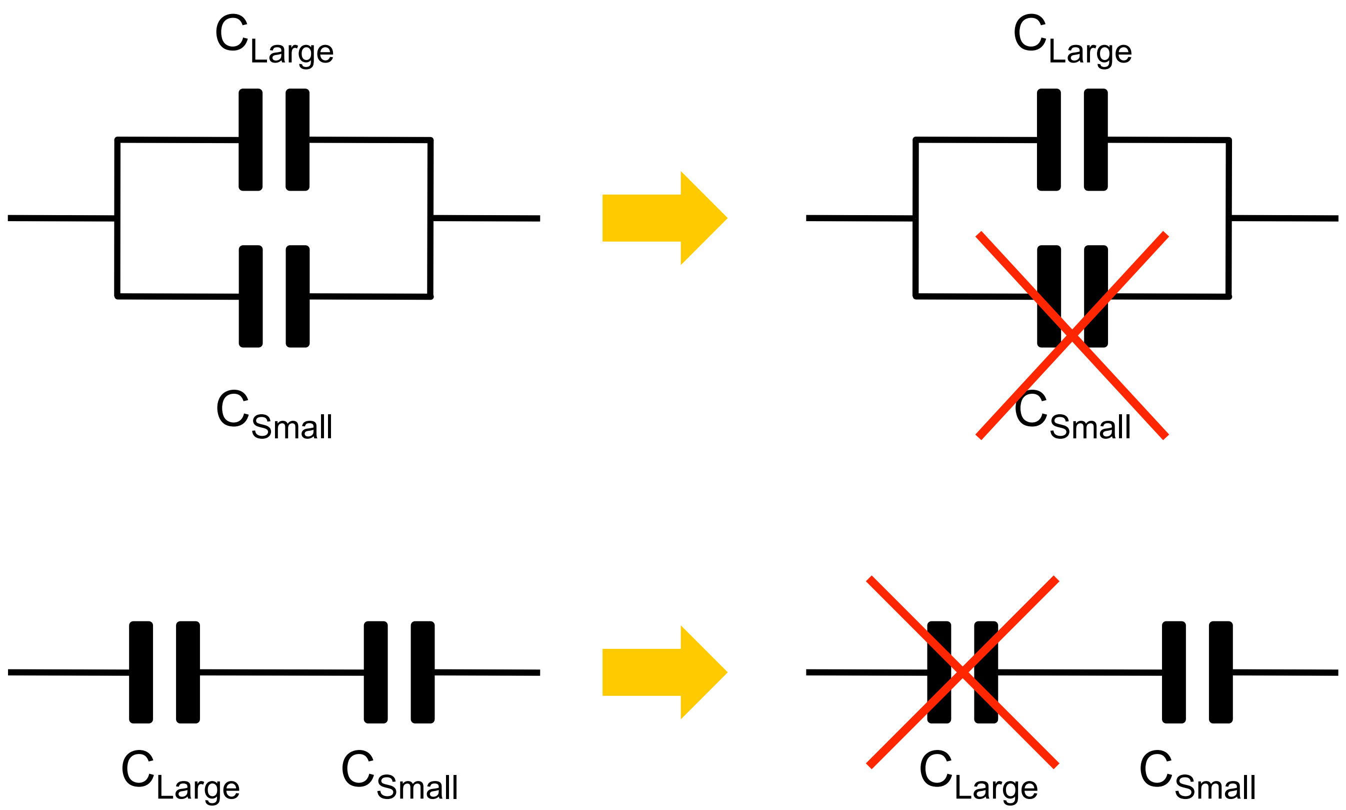
Large and Small Values

- To roughly understand behavior of circuits, only keep the dominant components:



- Eliminate *larger* or the *smaller* part (depending on circuit!)
- Error ~ ratio of components

The same for Capacitors



Resistors AND Capacitors

- Behavior depends on frequency   ($|Z_C| = 1/(2\pi\nu C)$)