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Exercise 1:
Thévenin Equivalent & RC-Circuits

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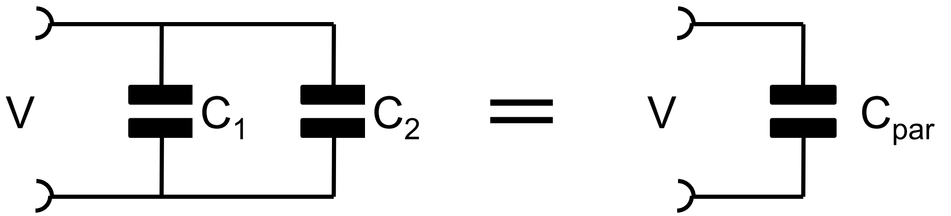
Recommendations

- I strongly recommend to use a mathematical program (Mathematica, Maple, SageMath,..) to solve the exercises

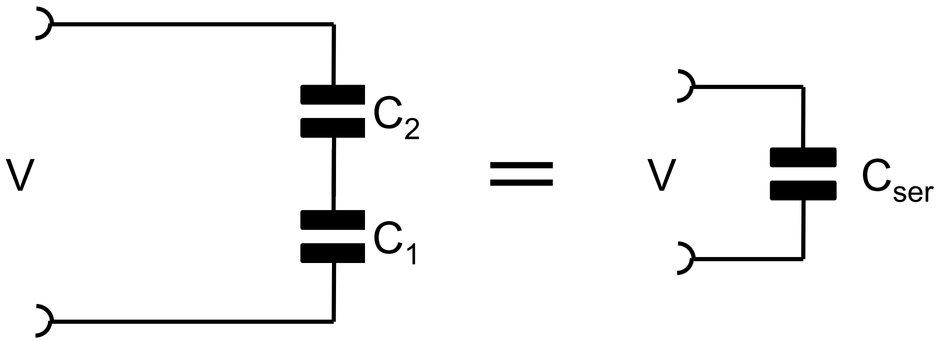
- Inspect each result:
  - What happens for $\omega \rightarrow 0,\infty$?
  - What happens if component values go to 0 or ∞?

Exercise 1.0

- Derive the expressions for the series and parallel connection of capacitors using
  - Charge conservation
  - Complex impedance & Kirchhoff's law

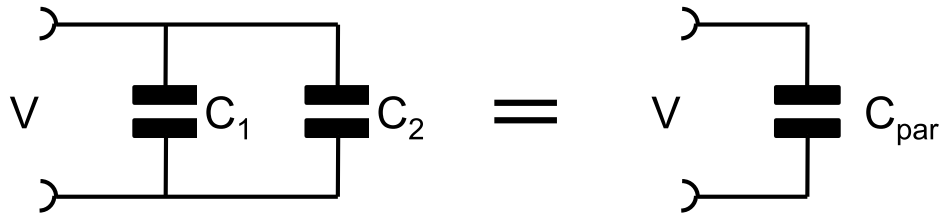


C_{par} = ???



C_{ser} = ???

Solution 1.0



C_{par} = ???

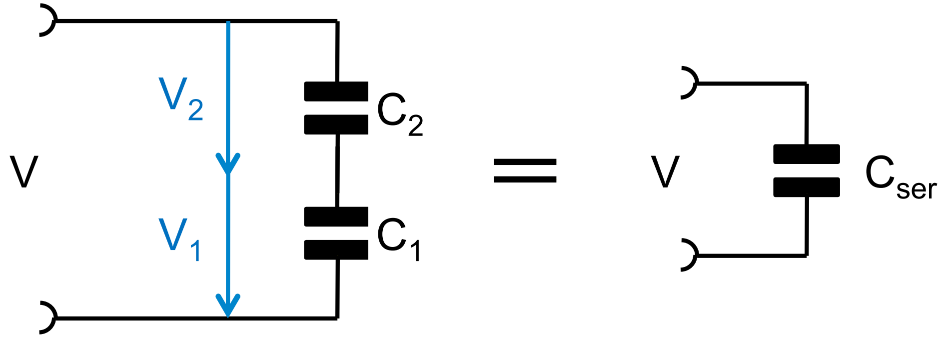
1. Charge conservation:

$$V \times C_1 + V \times C_2 = Q_1 + Q_2 = Q_{par} = V \times C_{par} \quad \rightarrow \quad C_1 + C_2 = C_{par}$$

2. Kirchhoff & complex impedance:

$$V\, sC_1 + V\, sC_2 = i_1 + i_2 = i_{par} = V\, sC_{par} \quad \rightarrow \quad C_1 + C_2 = C_{par}$$

Solution 1.0



C_{ser} = ???

1. Charge conservation:

Note: no charge can 'escape' the middle node, so that $Q_1=Q_2=Q_{ser}$

$V = V_1 + V_2 = Q_1/C_1 + Q_2/C_2 = Q/C_1 + Q/C_2 = Q/C_{ser}$

$\rightarrow \quad 1/C_1 + 1/C_2 = 1/C_{ser}$

2. Kirchhoff & complex impedance:

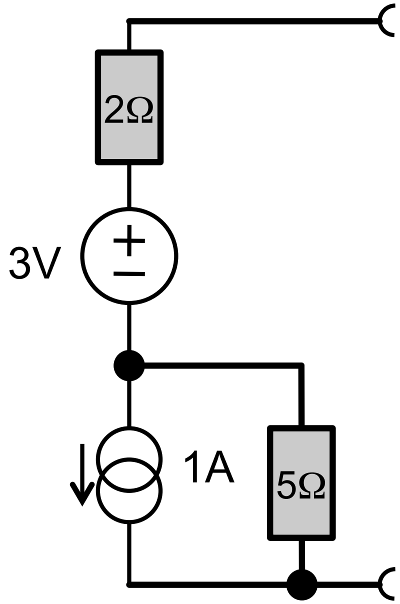
$V_1\, sC_1 = V_2\, sC_2$ and $V_1 + V_2 = V \quad \rightarrow \quad V_1 = V\, C_2 / (C_1+C_2)$

$\rightarrow \quad i_1 = V_1\, sC_1 = V\, s\, C_1C_2 / (C_1+C_2)$

$\rightarrow \quad C_{ser} = i / (Vs) = i_1 / (Vs) = C_1C_2 / (C_1+C_2)$

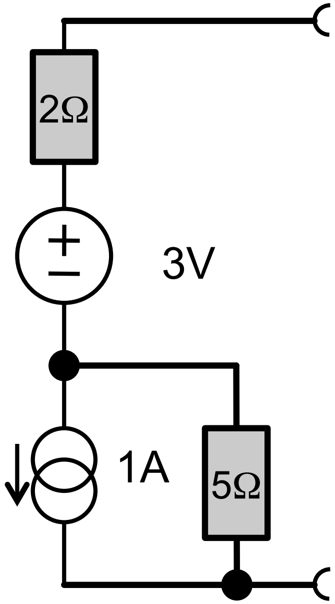
Exercise 1.1

- Derive the Thévenin Equivalent for the following circuit:



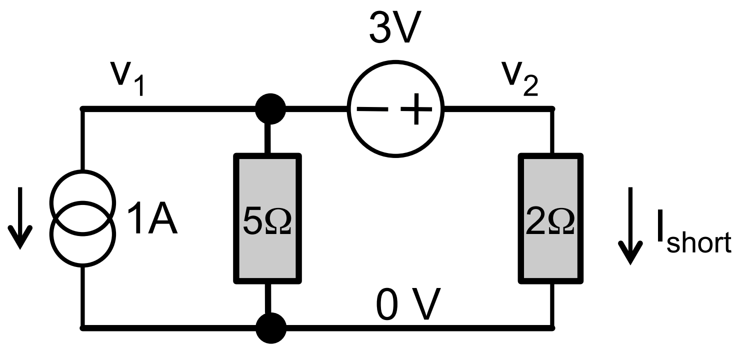
- Try two different methods:
  - Use the Open/Short method with Kirchhoff's rules
  - Convert the I-source part to a voltage source first…

Solution 1.1 – Kirchhoff



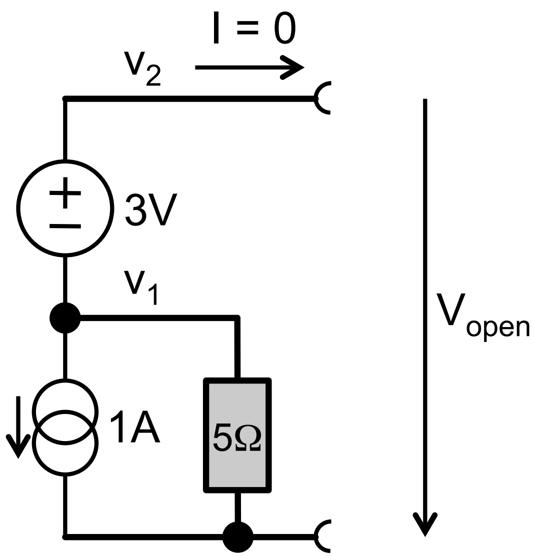
1. Short circuit current:

- EQ1: $1\ A + v_1 / 5\Omega + v_2 / 2\Omega = 0$
- EQ2: $v_2 = v_1 + 3V$
- $\rightarrow v_2 = -4/7\ V$
- $\rightarrow I_{short} = -2/7\ A$



2. Open circuit voltage:

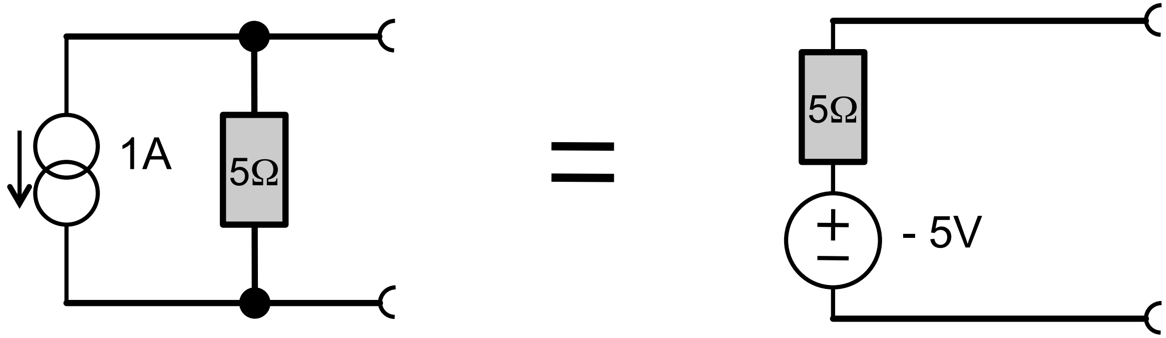
- EQ1: $1\ A + v_1 / 5\Omega = 0$
- EQ2: $v_2 = v_1 + 3V$
- $\rightarrow v_1 = -5\ V$
- $\rightarrow v_2 = V_{open} = -2\ V$



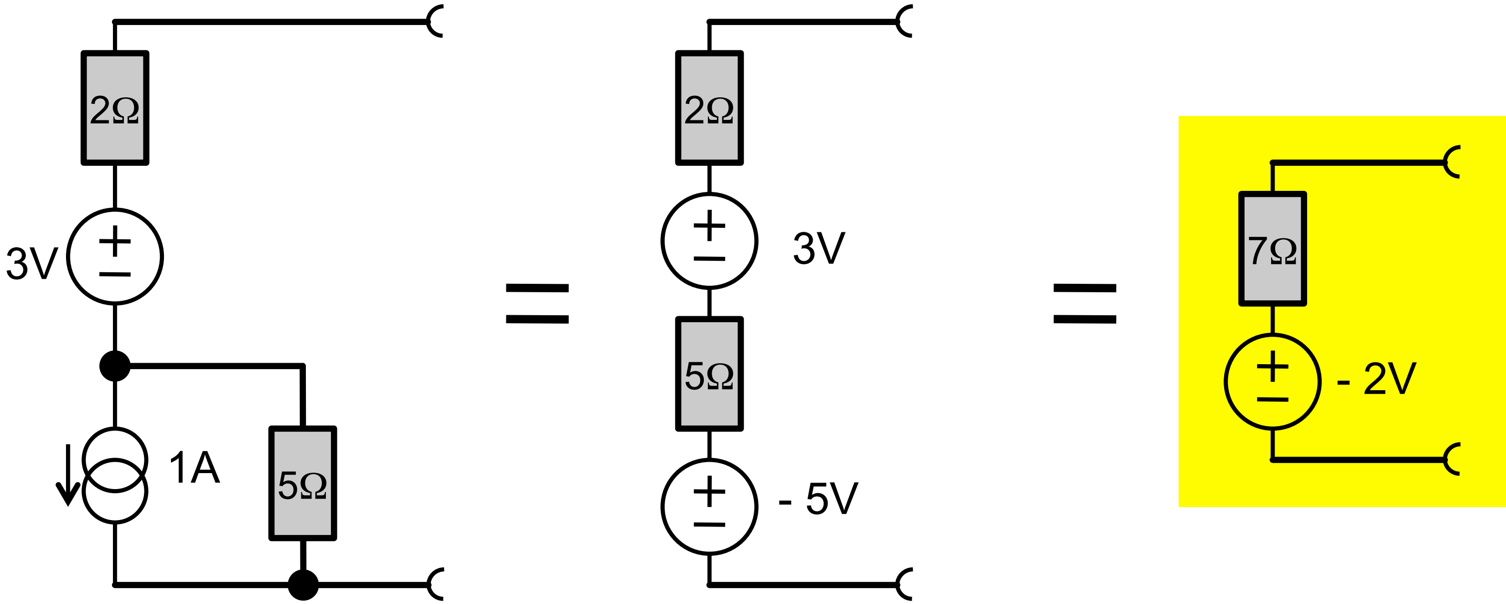
- Source: $V_0 = V_{open} = -2\ V$, $R_V = V_0 / I_{short} = 7\ \Omega$

Solution 1.1 – Thévenin Transformations

1. Convert the current source to a voltage source:

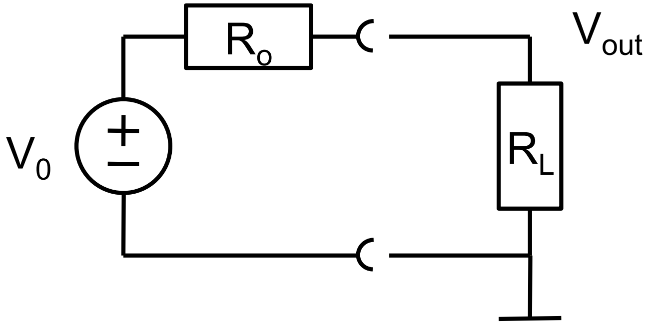


2. Use this in the circuit:



Exercise 1.2

- A voltage source with voltage V_0 and output resistance R_0 is loaded by a resistor R_L:



- What is the output voltage V_{out} ?
- Which current flows in R_L ?
- What power is dissipated in R_L ?
  - Check that noting is dissipated for R_L=0 and $R_L \rightarrow \infty$
- For which value of R_L is the dissipation maximized?
  - What is the dissipation?

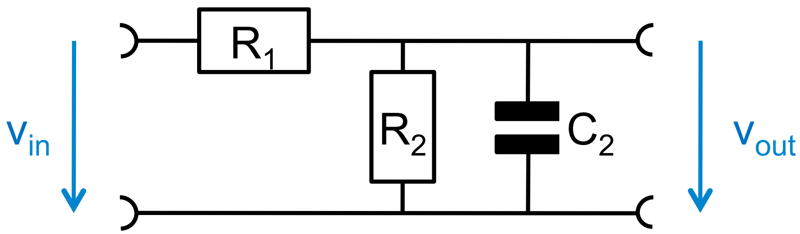
Solution 1.2

```
In[29]:= Vout = V0 RL/(R0 + RL);

In[30]:= Iout = Vout/RL

Out[30]= V0/(R0 + RL)

In[31]:= Pout = Vout Iout

Out[31]= RL V0^2/(R0 + RL)^2

In[38]:= Table[Limit[Pout, RL → x], {x, {0, ∞}}]

Out[38]= {0, 0}

In[39]:= Solve[D[Pout, RL] == 0, RL] // First

Out[39]= {RL → R0}

In[40]:= Pout /. %

Out[40]= V0^2/(4 R0)
```

$$\text{Vout} = \text{V0}\frac{\text{RL}}{\text{R0}+\text{RL}}, \quad \text{Iout} = \frac{\text{Vout}}{\text{RL}} = \frac{\text{V0}}{\text{R0}+\text{RL}}, \quad \text{Pout} = \frac{\text{RL}\,\text{V0}^2}{(\text{R0}+\text{RL})^2}, \quad \text{Pout}|_{\text{RL}=\text{R0}} = \frac{\text{V0}^2}{4\,\text{R0}}$$

Exercise 1.3

- Derive the Transfer Function of this circuit:



- Use 3 different approaches:
  - Treat the circuit directly (using Kirchhoff's rule)
  - Consider it as a voltage divider of two Impedances. Use R_1 for Z_1 and the parallel connection of R_2 and C_2 for Z_2
  - Replace the (resistive) voltage divider by its Thévenin equivalent and then add the capacitor

- Make a Bode Plot
  - Describe the difference to the normal Low Pass Filter

Solution 1.3

Direct Treatment:

```
EQ = (Vin - Vout)/R1 == Vout s C2 + Vout/R2;

Solve[EQ, Vout] // First
```

$$\left\{ \text{Vout} \to \frac{\text{R2 Vin}}{\text{R1} + \text{R2} + \text{C2 R1 R2 s}} \right\}$$

```
Hdirect = Vout/Vin /. %
```

$$\frac{\text{R2}}{\text{R1} + \text{R2} + \text{C2 R1 R2 s}}$$

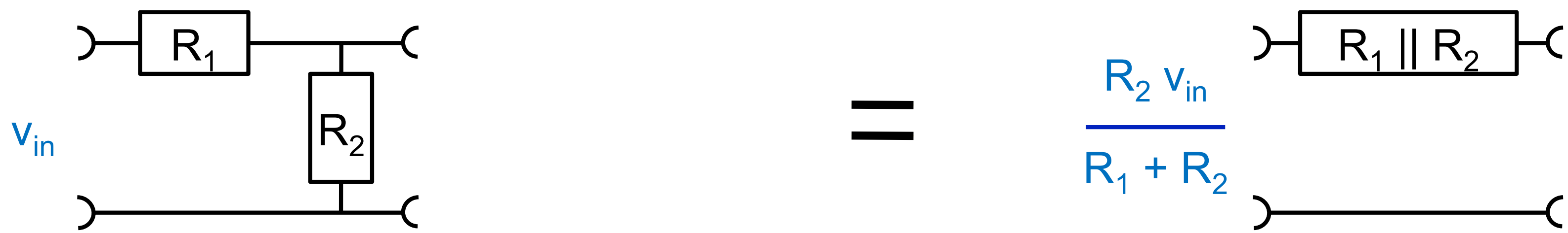
Voltage Divider:

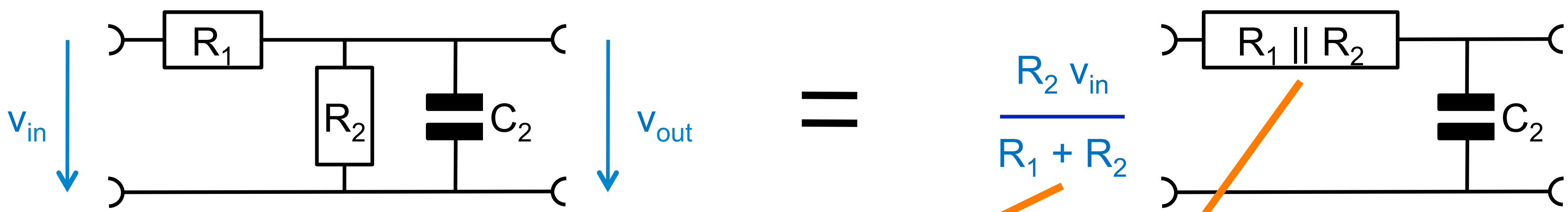
```
Hdiv = Z2/(Z1 + Z2) /. {Z1 → R1, Z2 → (1/R2 + s C2)^-1} // Simplify
```

$$\frac{\text{R2}}{\text{R1} + \text{R2} + \text{C2 R1 R2 s}}$$

```
Hdirect == Hdiv

True
```

Solution 1.3





$$\text{Hthenevin} = \frac{g}{1 + s\,RR\,C2} \;/.\; \left\{ g \to \frac{R2}{R1 + R2},\; RR \to \left(\frac{1}{R1} + \frac{1}{R2}\right)^{-1} \right\} \;//\; \text{Simplify}$$

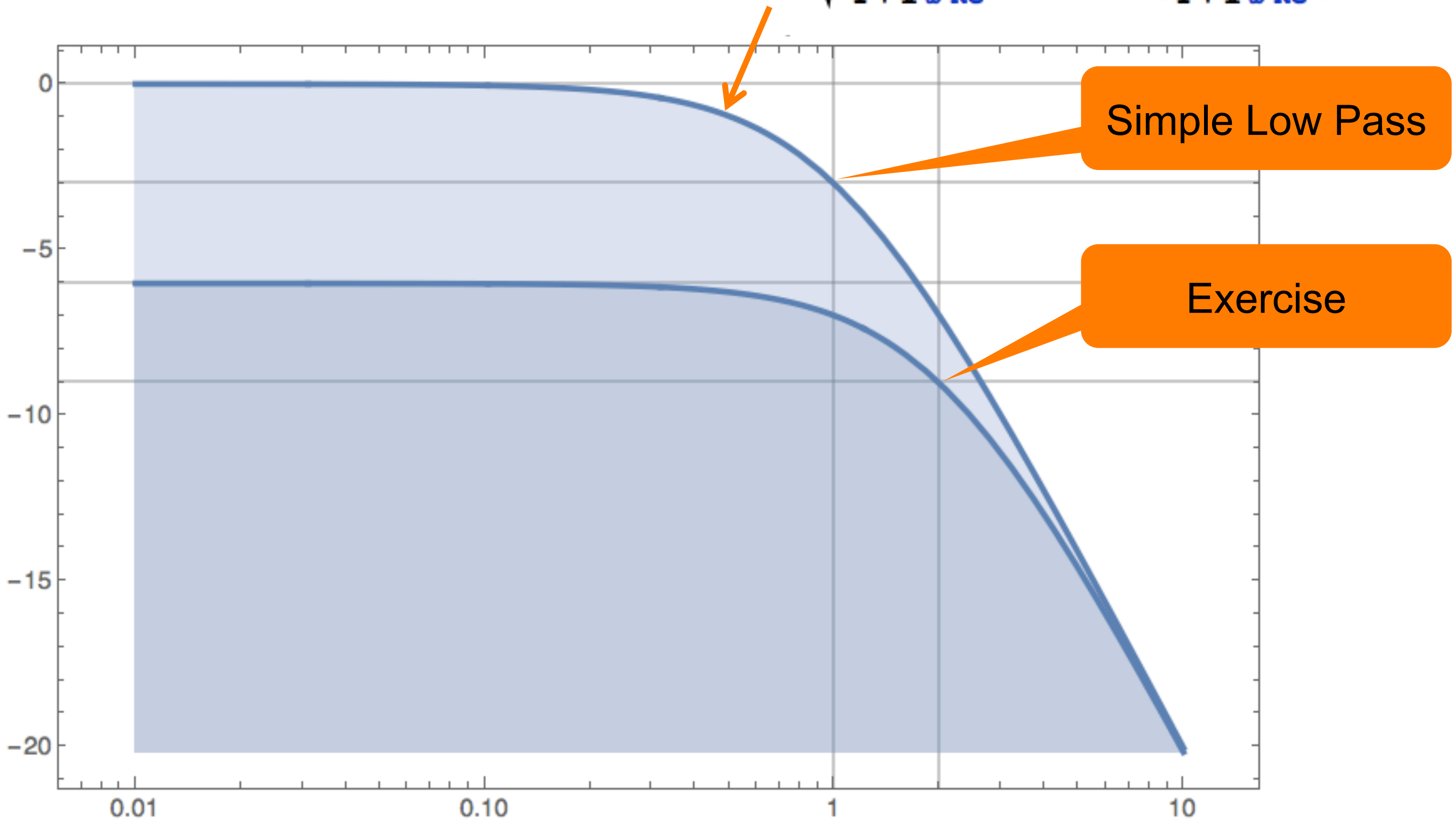
$$\frac{R2}{R1 + R2 + C2\,R1\,R2\,s}$$

Simple Low Pass

Solution 1.3

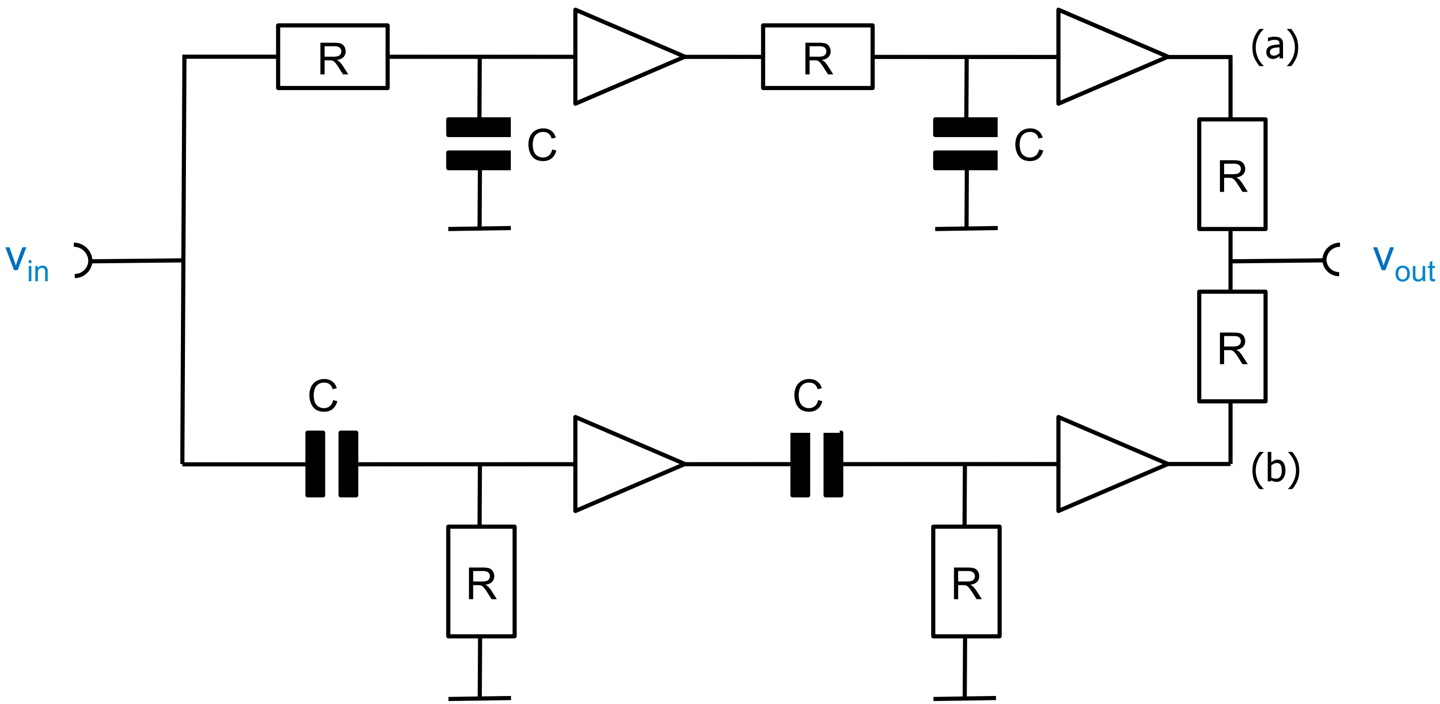
- Compared to the ‘simple’ Low-Pass:
  - The signal is attenuated by $R_1/(R_1+R_2)$
  - The time constant is lowered (i.e. the corner frequency is raised)
- Plot for $R_1 = R_2 = C_2 = 1$:

$$\text{HLP} = \sqrt{\frac{1}{1 + i\,\omega\,\text{RC}}\ \text{Conjugate}\left[\frac{1}{1 + i\,\omega\,\text{RC}}\right]}\ /.\ \{\text{RC} \to 1\}$$



Exercise 1.4: Notch Filter

- Consider the following circuit made of cascaded High- and Low Pass stages:
  - The resistors at the output just add the signals at (a) and (b)



- What is the output signal at the corner frequency?
  - Explain this by comparing amplitudes ***and phases*** at (a) and (b)

Solution 1.4

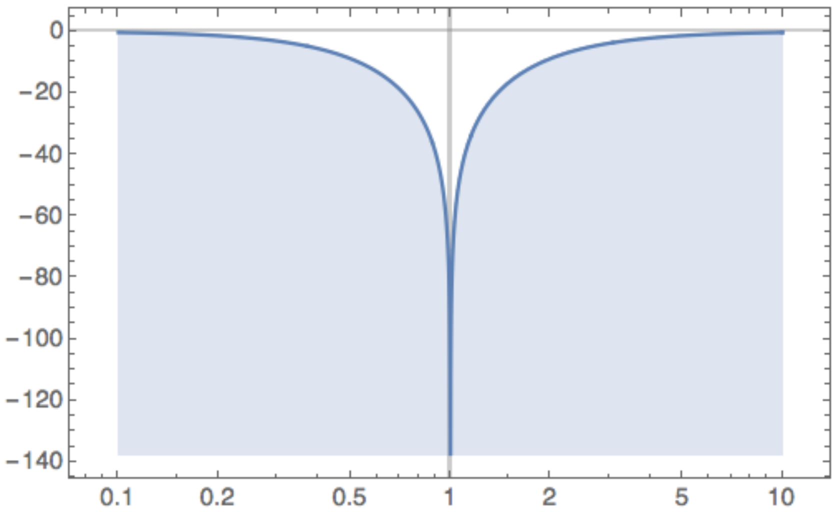
```
$Assumptions = ω > 0 && RC > 0; HLP = 1/(1 + ⅈ ω RC); HHP = ⅈ ω RC/(1 + ⅈ ω RC);

H = HLP HLP + HHP HHP // Simplify
```

$$\frac{-1 + RC^2\,\omega^2}{(-i + RC\,\omega)^2}$$

```
HMag = H Conjugate[H] /. RC → 1 // FullSimplify
```

$$\frac{\left(-1 + \omega^2\right)^2}{\left(1 + \omega^2\right)^2}$$

```
LogLinearPlot[dB[HMag], {ω, 0.1, 10}, GridLines → {{1}, {0}}]
```



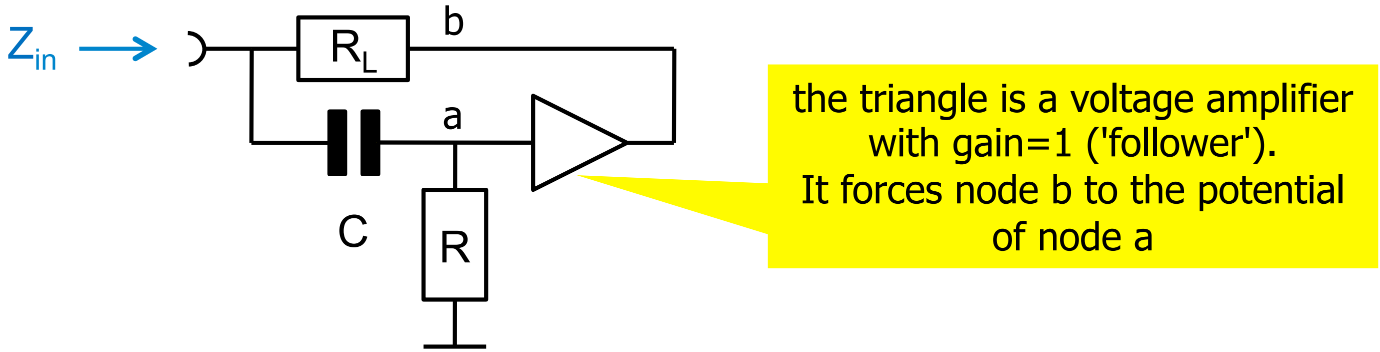
- At the corner frequency, the signal is fully stopped!
- This is because **the phases** of the two signals are ± 90°, i.e. the signals are complementary. For the HP:

```
Limit[ArcTan[Im[HLP HLP]/Re[HLP HLP] /. {RC → 1}], ω → 1] // ComplexExpand
```

$$\frac{\pi}{2}$$

Exercise 1.5: Gyrator

- A 'Gyrator' can mimic inductive behaviour, while using only resistors, capacitors and amplifiers
- Consider the following circuit:

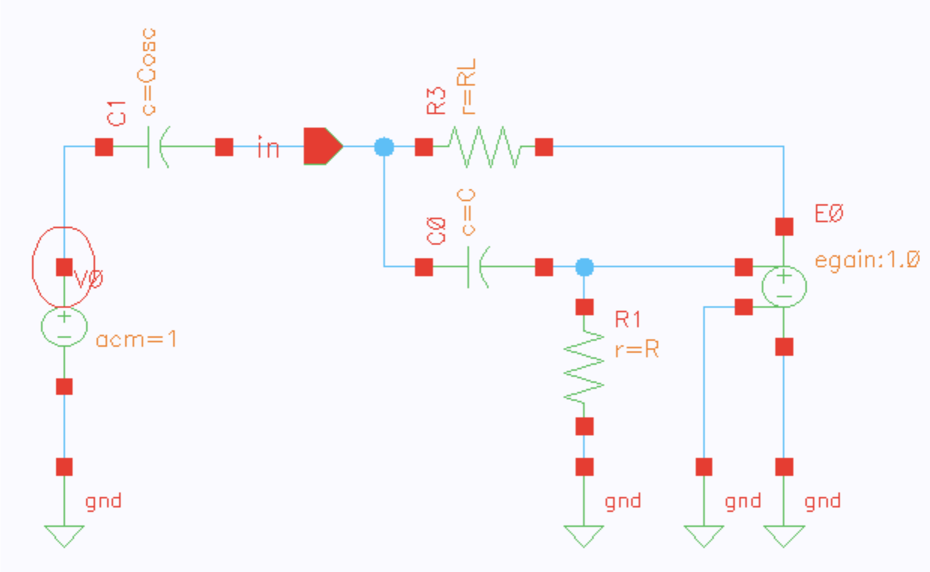


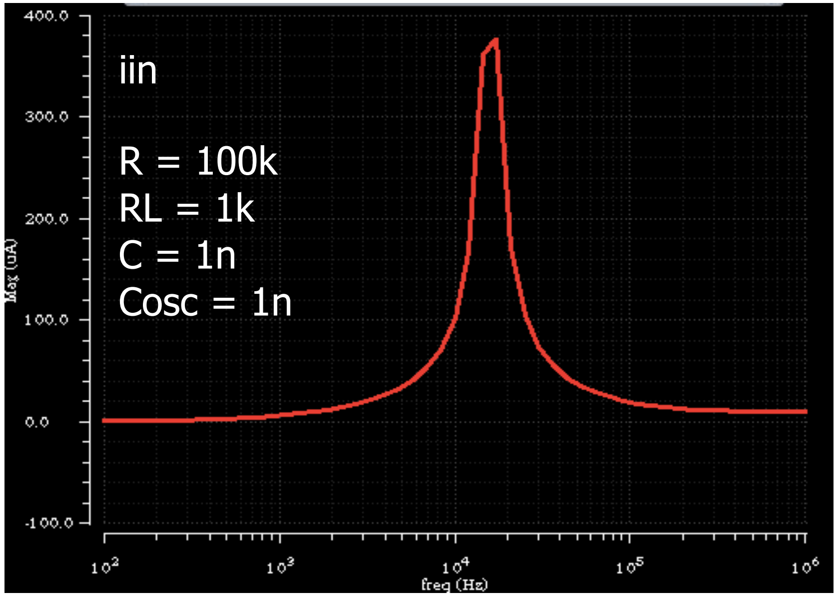
- **Calculate** the input impedance $Z_{in} = U_{in}/I_{in}$ of the circuit
  - (Use Kirchhoff's law at the input node and node a)
- For frequencies < $1/C\ R_L$, the denominator can be neglected.
- Compare the result to an inductor in series with R_L
- Simulate.
  - Note that R should be larger than R_L (what happens for $R=R_L$?)
  - Plot i_{in}.
  - Add another capacitor in series to produce a resonant circuit.

Solution 1.5

- Mathematica:

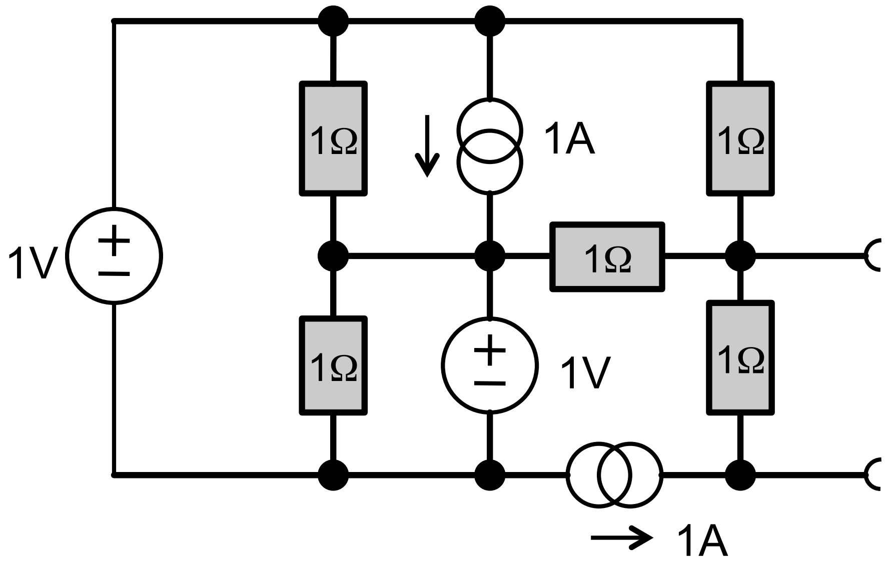
```
EQin = iin == (vin - va) s C + (vin - vb)/RL /. vb → va;
EQa = (vin - va) s C == va / R;

Eliminate[{EQin, EQa}, va] // Simplify
iin (RL + C R RL s) == vin + C RL s vin

sol = Solve[%, iin] // First
{iin → (vin + C RL s vin)/(RL (1 + C R s))}

Zgyrator[s_] = vin/iin /. sol // Simplify
(RL + C R RL s)/(1 + C RL s)
```



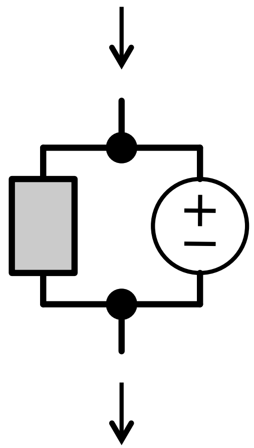


Exercise 1.2

- Derive the Thévenin Equivalent for the following circuit:

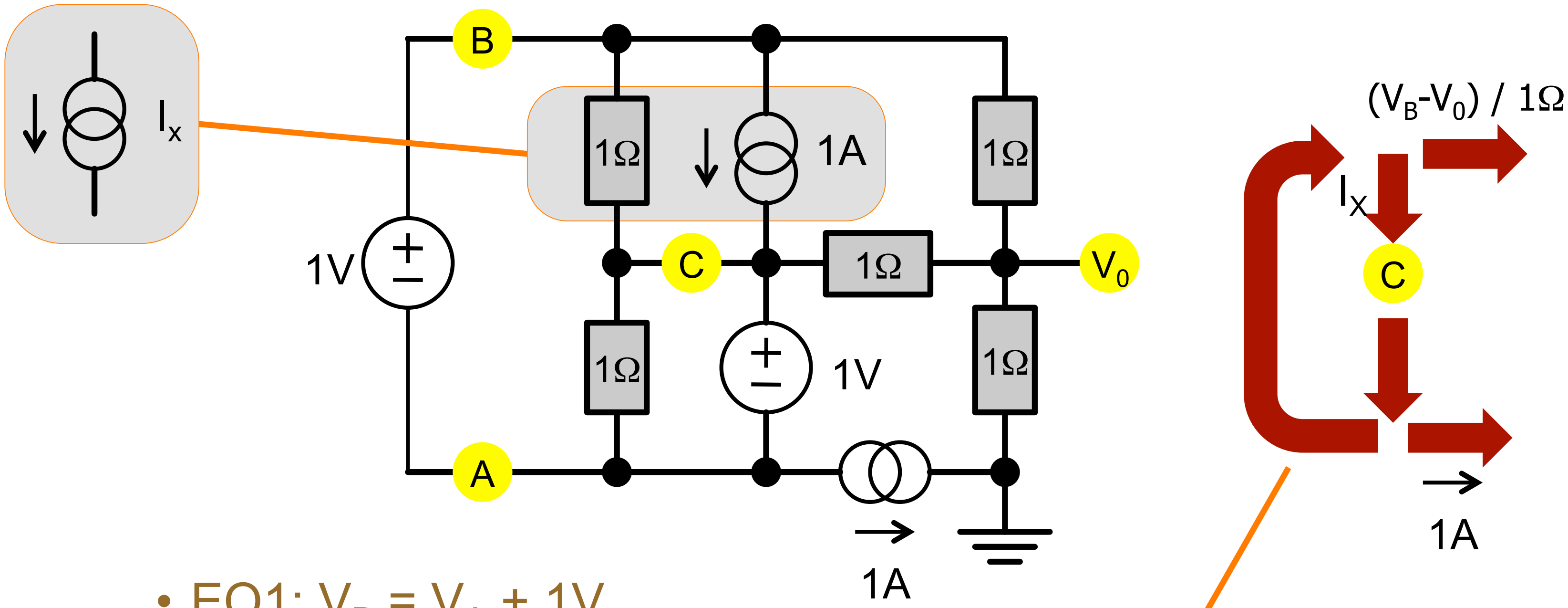


- Hint:
  - Consider the effect of a fixed voltage source here:
  - The current is 'passed through' (because the voltage on R cannot change!)



Solution 1.2

- Open circuit voltage (4 nodes → we need 4 equations)

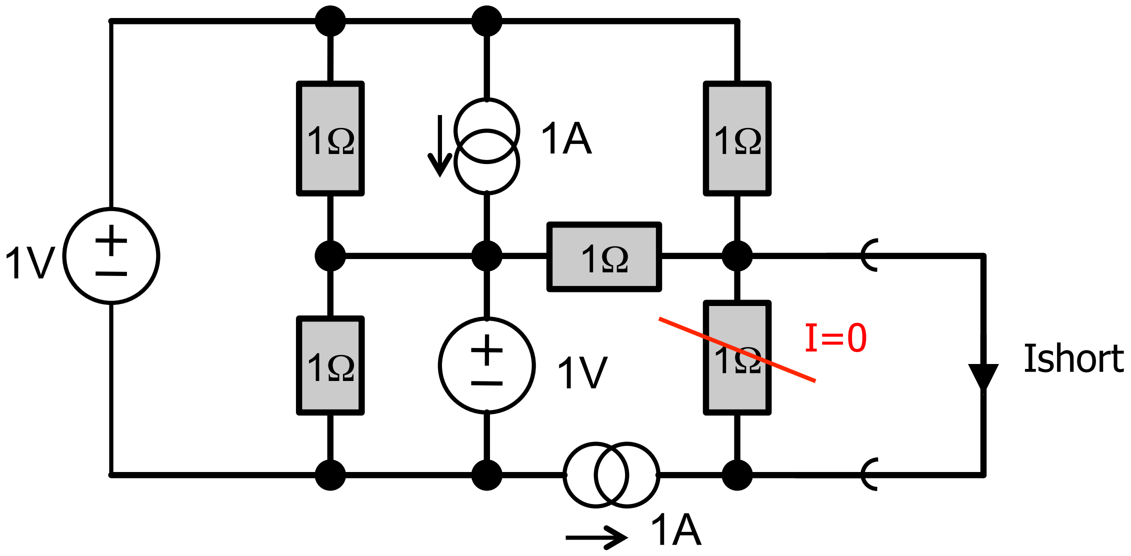


- EQ1: $V_B = V_A + 1V$
- EQ2: $V_C = V_A + 1V$
- EQ3: $(V_B-V_0) / 1\Omega + (V_C-V_0) / 1\Omega = V_0 / 1\Omega$ (current sum at V_0)
- EQ4: $I_x = (V_C-V_0) / 1\Omega + 1A + I_x + (V_B-V_0) / 1\Omega$ (current sum at V_C)

→ $V_0 = -1V$ ($V_A = -2.5$ V, $V_B = V_C = -1.5$ V)

Solution 1.2

- Short circuit current:
  - This is simple due to the particular circuit: For a short, the voltage at the output – resistor is 0 V → the current in the resistor is 0 A
  - I_{short} = - 1A is fixed by the lower current source



- Thévenin Equivalent:
  - V_0 = -1V
  - $R_V = V_0 / I_{short}$ = 1Ω