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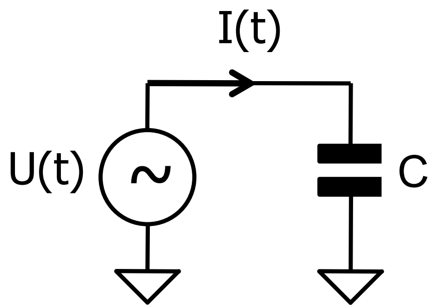
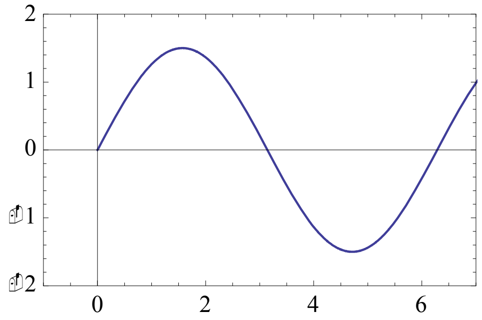
AC BEHAVIOR OF COMPONENTS

AC Behavior of Capacitor

- Consider a capacitor driven by a sine wave voltage:

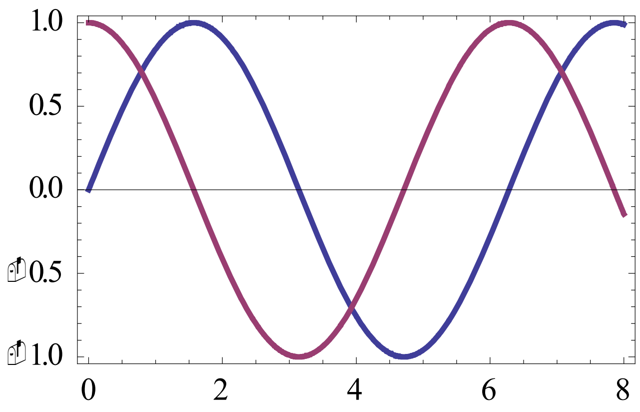
$$U(t) = U_0 \sin(\omega t + \varphi)$$





- The current: $I(t) = C \dfrac{dU(t)}{dt} = C\, U_0\, \omega \cos(\omega t + \varphi)$

  is shifted by 90° (sin ↔ cos)!



Complex Impedance

- To simplify our calculations, we would like to extend the relation R= U/I to capacitors, using an **impedance** Z_C .

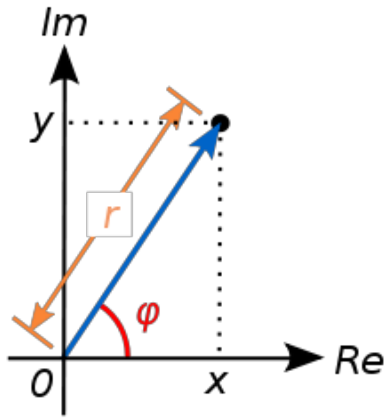
- In order to get the **phase** right, we use **complex** quantities:

$$U(t) = U_0 \sin(\omega t + \varphi) \quad \rightsquigarrow \quad U_0 \cdot e^{i(\omega t+\varphi)} = U_0[\cos(\omega t+\varphi)+i\sin(\omega t+\varphi)]$$

  for voltages and currents.
  - By mixing complex and real parts, we can mix sin() and cos() components and therefore influence the phase.

- Note: Often 'j' is used instead of 'i' for the complex unit, because 'i' is used as current symbol…

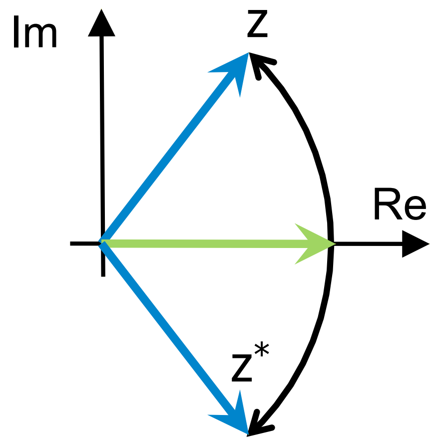
- **Often 's' is used for iω (or jω)**



From Complex Values back to **Real** Quantities

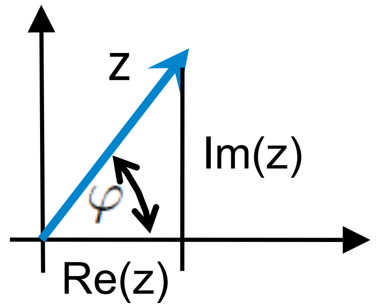
- To find ('back') the **amplitude** of such a complex signal, we calculate the length (**magnitude**) of the complex vector as

$$a = \sqrt{zz^*}$$



- To get the **phase**, we use real and imaginary parts:

$$\varphi = \text{atan}\left(\frac{\text{Im(z)}}{\text{Re(z)}}\right)$$



Note: this simple formula works only in 2 quadrants. You may have to look at the signs of Re(z) and Im(z)

Hints for Mathematica

- Mathematica knows complex arithmetic
- Useful Functions are **`Abs[]`** and **`Arg[]`**
  - Remember: Imaginary Unit is typed as **`ESC i i ESC`**
- If you want to simplify expression, Math. has to **know** that expressions like ω, R, C, U are **real**.
  - This can be done with Assumptions:
  - Sometimes **`ComplexExpand[]`** can be used. It assumes all arguments are real (but not necessarily > 0):

```
$Assumptions = True;

{Abs[ⅈ ω], Arg[ⅈ ω]}

{Abs[ω], Arg[ⅈ ω]}

{Abs[ⅈ ω], Arg[ⅈ ω]} // FullSimplify

{Abs[ω], Arg[ⅈ ω]}

$Assumptions = ω > 0;

{Abs[ⅈ ω], Arg[ⅈ ω]} // Simplify

{ω, π/2}
```

```
{Abs[ⅈ ω], Arg[ⅈ ω]} // ComplexExpand

{Sqrt[ω^2], Arg[ⅈ ω]}
```

Complex Impedance of the Capacitor

- We know that

$$I(t) = C\,\frac{dU(t)}{dt}$$

- With $\quad U(t) = U_0 \cdot e^{i(\omega t + \varphi)}$

  we have $\quad I(t) = CU'(t) = C \cdot U_0 \cdot i\omega \cdot e^{i(\omega t + \varphi)}$

- Therefore

$$Z_C = \frac{U(t)}{I(t)} = \frac{1}{i\omega C} = \frac{1}{sC}$$

The impedance of a capacitor becomes very small at high frequencies

- Similar: $\quad Z_L = i\,\omega L = s\,L$

Checking this again for a Capacitor

- For an input voltage (sine wave of freq. ω) with phase = 0

$$U(t) = U_0\, e^{i\omega t}$$

we have

$$I(t) = \frac{U(t)}{Z_C} = U_0\, e^{i\omega t} \cdot i\omega C$$

- The amplitude of I(t) is

$$\begin{aligned} |I| &= \sqrt{I(t) I^*(t)} \\ &= \sqrt{U_0\, e^{i\omega t} \cdot i\omega C \times U_0\, e^{-i\omega t} \cdot (-i)\omega C} \\ &= \sqrt{U_0^2\, e^{i\omega t} e^{-i\omega t} \cdot (i\omega C)(-i\omega C)} \\ &= U_0 \omega\, C \end{aligned}$$

- The phase is:

$$\varphi = \operatorname{atan}\left(\frac{\omega\, C}{0}\right) = \operatorname{atan}(\infty) = \frac{\pi}{2}$$

  - We have dropped the time variant part and the constant U_0

Simplifying even more

- As we have just seen, the $U(t) = U_0 e^{i\omega t}$ propagates trivially to the output.
- We therefore drop this part and just use '1'!

Recipe to Calculate Transfer Functions

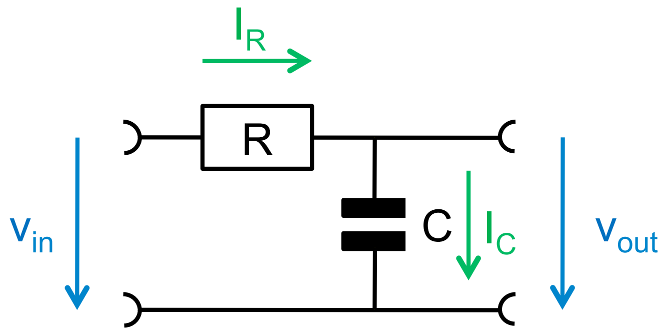
- Replace all component by their complex impedances (1/(sC), sL, R)
- Assume a unit signal of ‘1’ at the input

  (in reality it is $U(t) = U_0 \, e^{i\omega t}$)

- Write down all node current equations or current equalities using Kirchhoff’s Law (they depend on s)
  - You need N equations for N unknowns
- Solve for the quantity you are interested in (most often V_{out})
- Analyze the result (amplitude / phase / …)

Example: Low Pass

- Consider



- We have only *one* unknown: v_{out}

- Current equality at node v_{out}: $\frac{v_{in} - v_{out}}{R} = I_R = I_C = v_{out}\, s\, C$

- Solve for v_{out}:

$$\begin{aligned} v_{in} - v_{out} &= v_{out}\, s\, C\, R \\ v_{in} &= v_{out}(1 + s\, C\, R) \\ \frac{v_{out}}{v_{in}} &= H(s) = \frac{1}{1 + s\, C\, R} \end{aligned}$$

Mathematica Hint

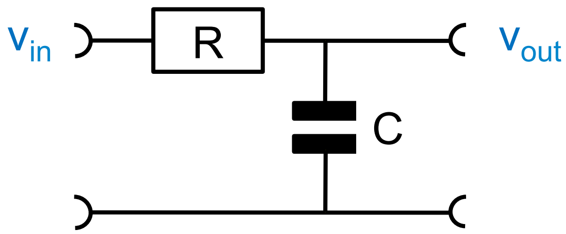
- Write down each node equation (here only 1):

```
EQ1 = (vin - vout)/R == vout s C;
```

- Solve them:

```
Solve[EQ1, vout] // First
```

$$\left\{ vout \to \frac{vin}{1 + C\,R\,s} \right\}$$

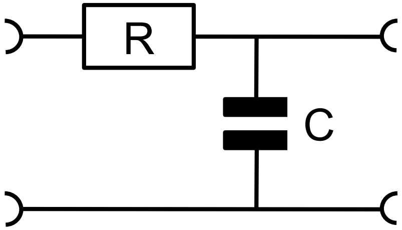


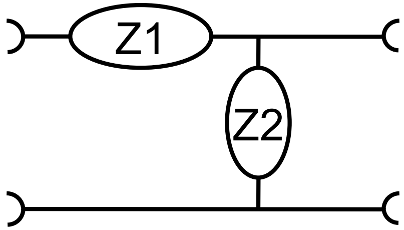
- Define a transfer function:

```
H[s_] = vout/vin /. %
```

$$\frac{1}{1 + C\,R\,s}$$

Low Pass as ‘complex’ voltage divider





- The LowPass can be seen as a ‘AC’ voltage divider with two impedances Z_1 = R and Z_2 = 1/sC

- Using the voltage divider formula, we get

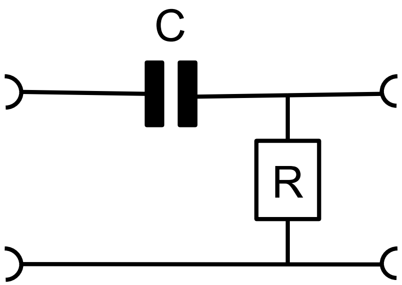
$$H(s) = \frac{v_{out}}{v_{in}} = \frac{Z_2}{Z_2 + Z_1} = \frac{\frac{1}{sC}}{\frac{1}{sC} + R} = \frac{1}{1 + s\,RC} = \frac{1}{1 + i\frac{\omega}{\omega_0}}$$

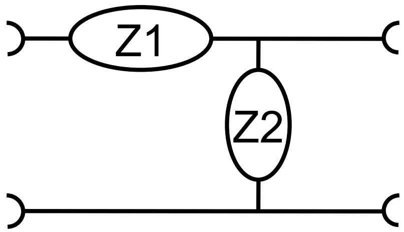
with ω_0 = 1/(RC), the ‘corner frequency’.

- This is the same as before…

The HIGH Pass

- By exchanging R and C, low frequencies are blocked and high frequencies pass through.

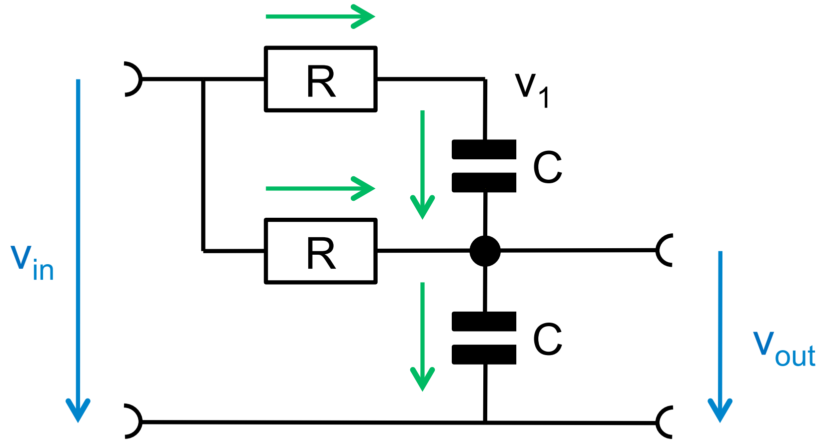




- We get $H_{HP}(s) = \frac{R}{R + \frac{1}{sC}} = \frac{sRC}{1 + sRC}$

- This is the (first order) 'High-Pass'.

A More Complicated Example



- We now have *two* unknowns: v_1, v_{out}

$$EQ1\,(@v_1) \quad : \quad \frac{v_{in} - v_1}{R} = (v_1 - v_{out})sC$$

$$EQ2\,(@v_{out}) \quad : \quad (v_1 - v_{out})sC + \frac{v_{in} - v_{out}}{R} = v_{out}\, sC$$

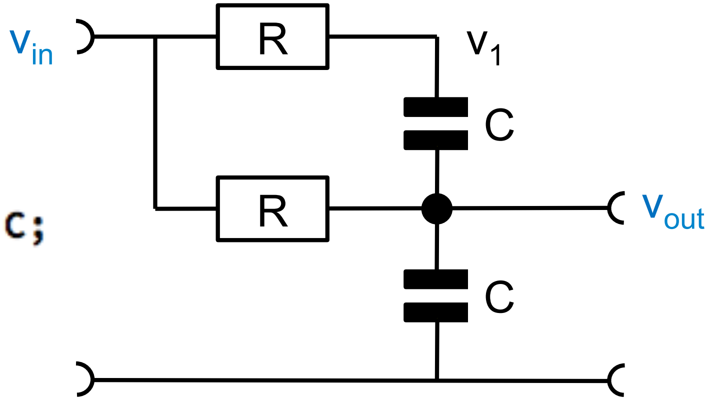
- Eliminating v_1 gives:

$$H(s) = \frac{1 + 2RC\, s}{1 + 3RC\, s + (RC)^2\, s^2}$$

- This is a *second order* TF. (order = max. exponent of s)

The same using Mathematica

- Node equations (here 2):




```
EQ1 = (vin - v1)/R == (v1 - vout) s C;
EQ2 = (vin - vout)/R + (v1 - vout) s C == vout s C;
```

- Solve them:

```
Solve[{EQ1, EQ2}, {vout, v1}] // First
```

$$\left\{vout \to -\frac{-vin - 2\,C\,R\,s\,vin}{1 + 3\,C\,R\,s + C^2\,R^2\,s^2},\; v1 \to \frac{(1 + 3\,C\,R\,s)\,vin}{1 + 3\,C\,R\,s + C^2\,R^2\,s^2}\right\}$$

- Define a transfer function:

```
H[s_] = vout/vin /. % // Simplify
```

$$\frac{1 + 2\,C\,R\,s}{1 + 3\,C\,R\,s + C^2\,R^2\,s^2}$$

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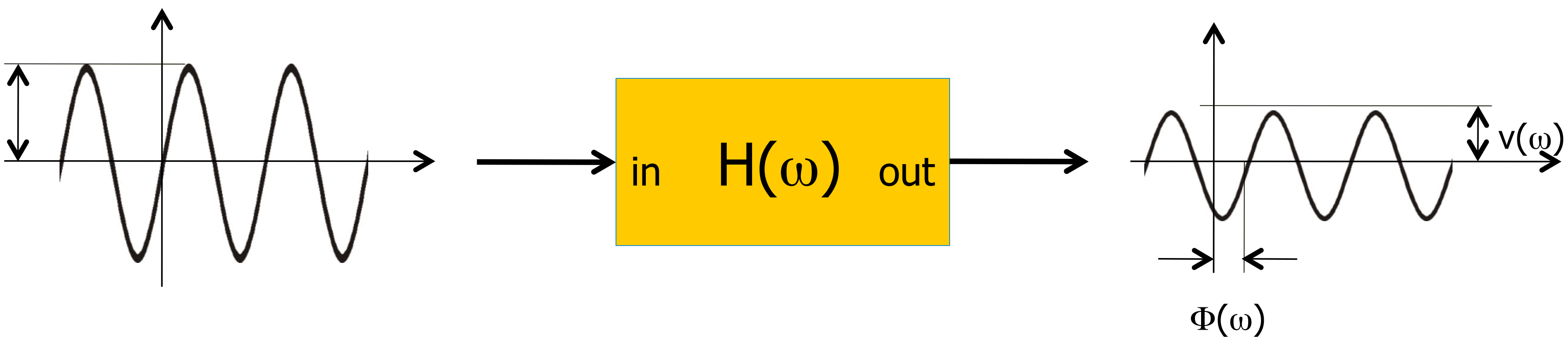
Bode Plot

Transfer Function

- The **transfer function** of a *linear, time invariant* system visualizes how the **amplitude** and **phase** of a **sine wave** input signal of **constant frequency** ω appears at the output

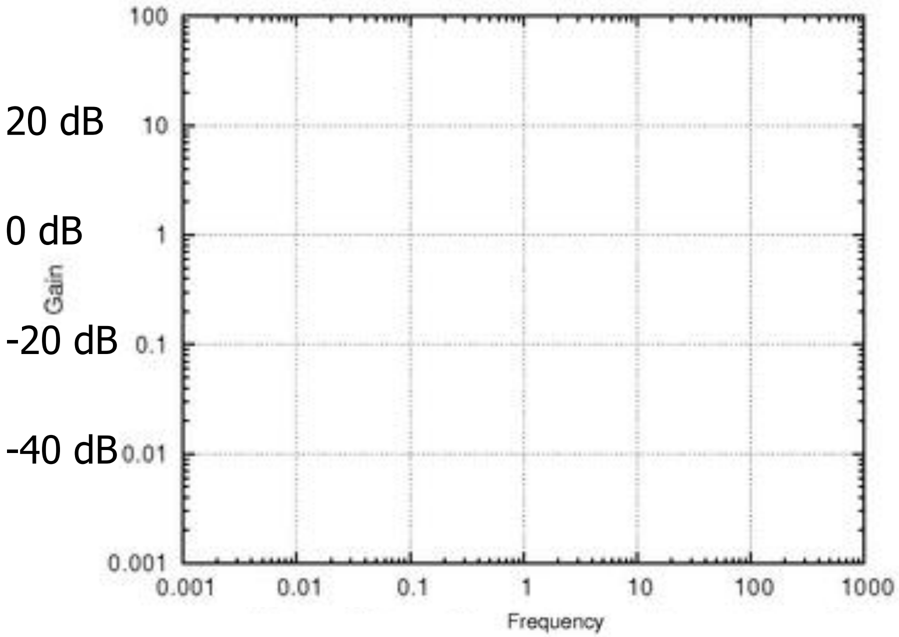
- The frequency remains unchanged
- The transfer function H(ω) contains
  - The phase change Φ(ω)
  - The gain v(ω) = amp_in / amp_out (ω)

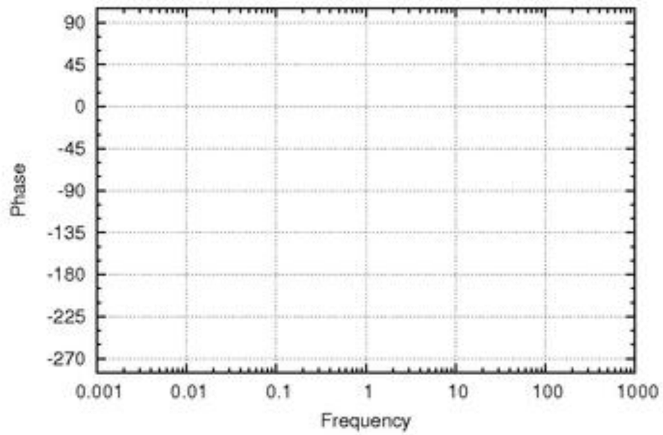


Bode Diagram: Definition

- The Bode Plot shows gain (+ phase) of the transfer function
- The frequency (x-axis) is plotted **logarithmically**
- Gain is plotted (y-axis) **logarithmically**, often in **decibel**
  - $DB(x) = 20 \log_{10}(x)$:

× 10	+20 dB	
× 100	+40 dB	
× 2	6 dB	(not exactly!)
× 1	0 dB	
/ 2	-6 dB	
/ √2	-3 dB	





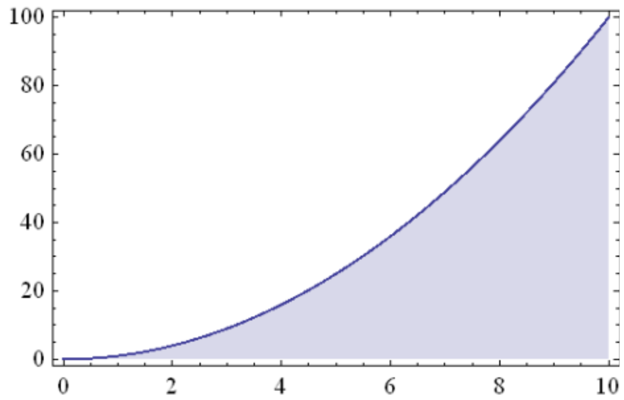
  - dBs for multiplied quantities just add !

Bode Diagram: Properties

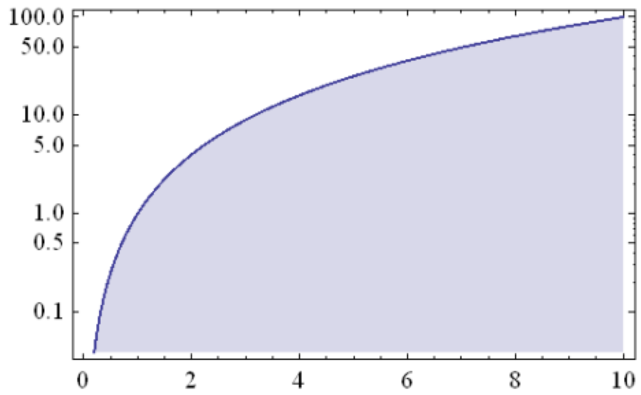
- Power functions are straight lines:

$$f(x) = x^n \Rightarrow \log[f(x)] = n \log(x)$$

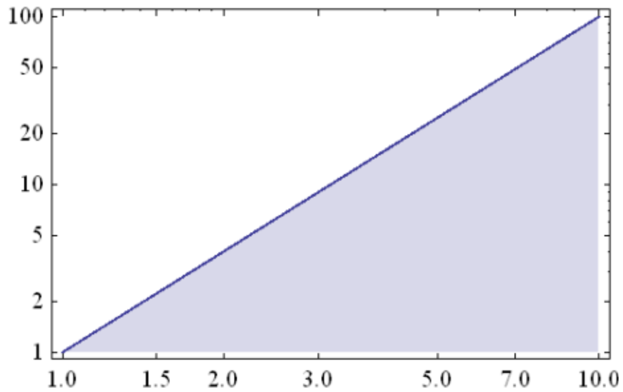
Plot[x²,{x,0,10}]



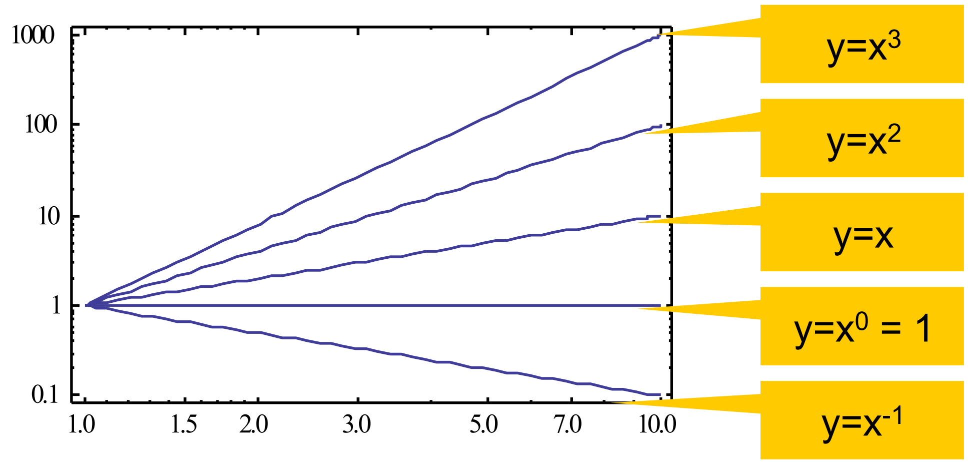
LogPlot[x²,{x,0,10}]



LogLogPlot[x²,{x,1,10}]



LogLogPlot[Table[x^N,{N,-1,3}],{x,1,10}]



Bode Diagram: Properties

- 1/x function has slope -1:

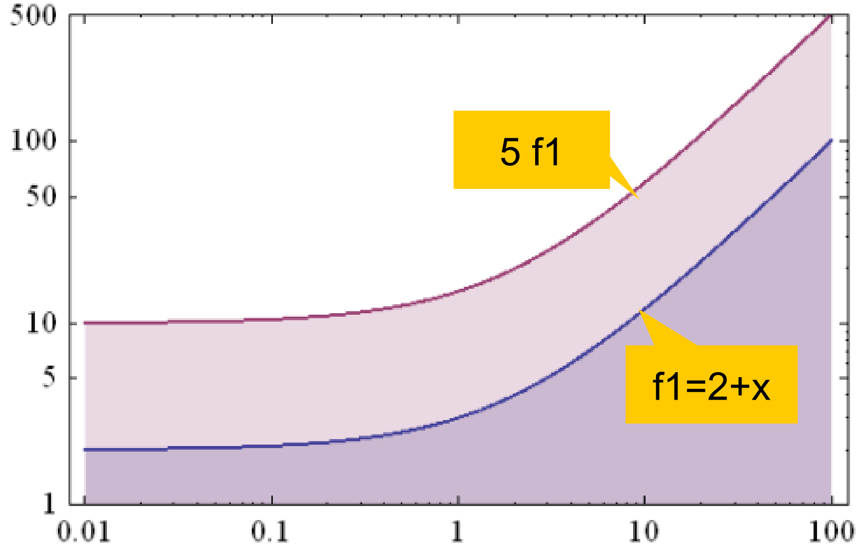
$$f(x) = \frac{1}{x} = x^{-1} \Rightarrow \log[f(x)] = -1 \log(x)$$

- Multiplied functions are **added** in plot:

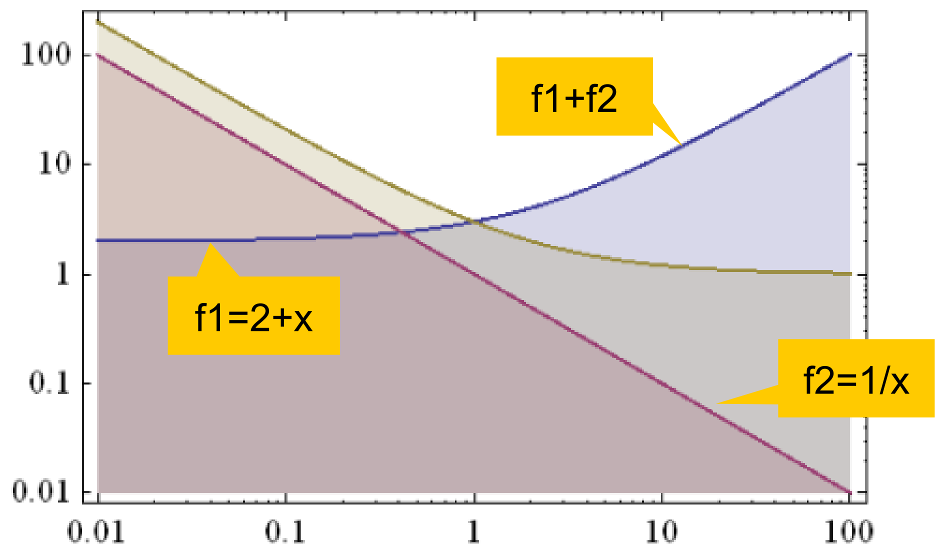
$$f = f_1 \cdot f_2 \Rightarrow \log[f] = \log(f_1) + \log(f_2)$$

f1=2+x;f2=x^{-1};

LogLogPlot[{f1, 5 f1},{x,0.01,100}]



LogLogPlot[{f1, f2,f1 * f2},{x,0.01,100}]



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THE LOW PASS FILTER

Analysis of the Low Pass Transfer Function

- Transfer Function: $$H(\omega) = \frac{1}{1 + i\frac{\omega}{\omega_0}}$$

- Magnitude: $$v(\omega) = \sqrt{H(\omega)H^*(\omega)} = \frac{1}{\sqrt{(1 + i\frac{\omega}{\omega_0})(1 - i\frac{\omega}{\omega_0})}}$$

$$v(\omega) = \frac{1}{\sqrt{(1 + \frac{\omega^2}{\omega_0^2})}}$$

$$\rightarrow 1 \quad \text{for} \quad \omega \rightarrow 0$$
$$\rightarrow \frac{1}{\sqrt{2}} \quad \text{for} \quad \omega = \omega_0$$
$$\rightarrow \frac{\omega_0}{\omega} \quad \text{for} \quad \omega \rightarrow \infty$$

- Phase: $$H(\omega) = \frac{1}{1 + i\frac{\omega}{\omega_0}} = \frac{1}{1 + i\frac{\omega}{\omega_0}} \times \frac{1 - i\frac{\omega}{\omega_0}}{1 - i\frac{\omega}{\omega_0}} = \frac{1 - i\frac{\omega}{\omega_0}}{1 + \frac{\omega^2}{\omega_0^2}}$$

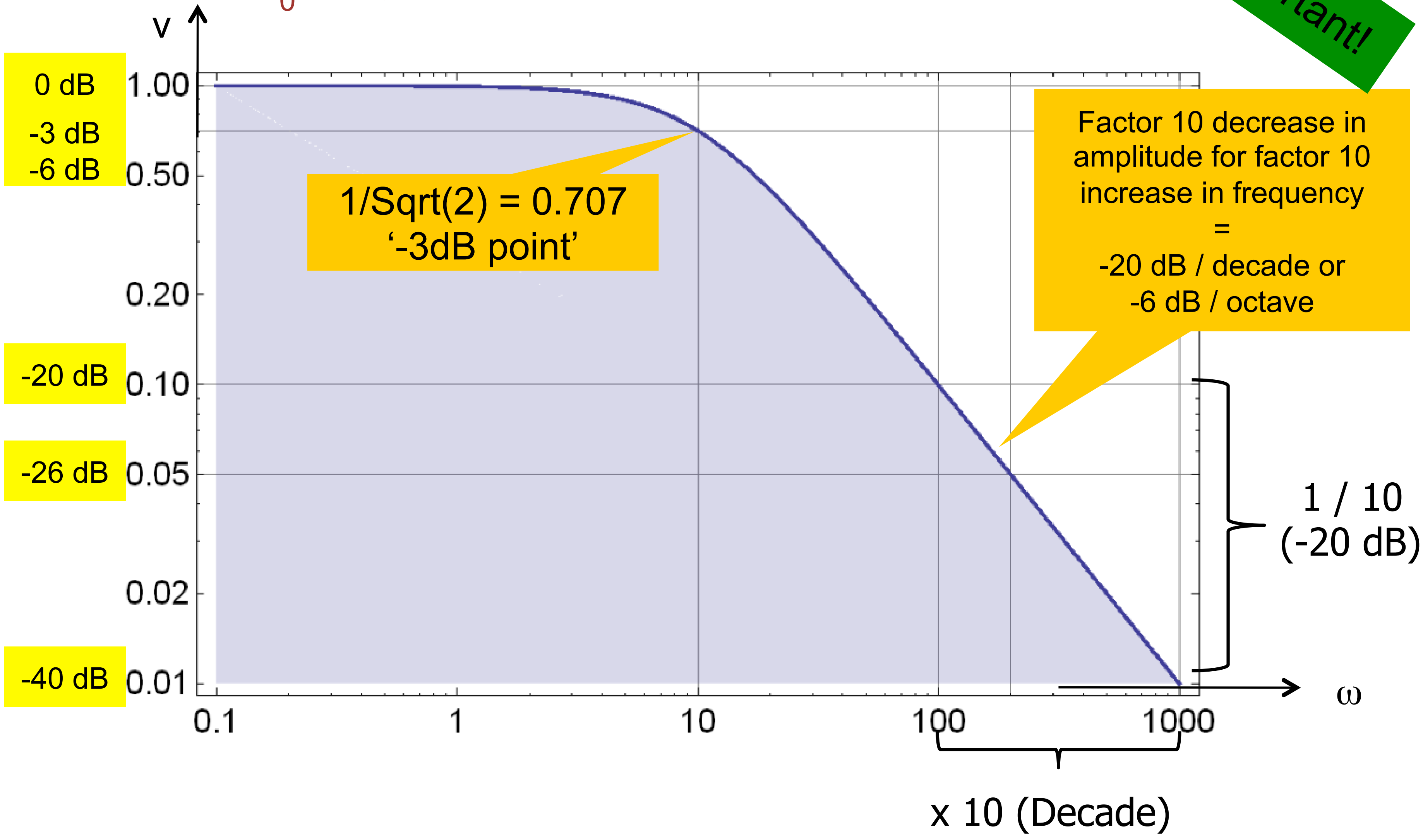
$$\varphi = \text{atan}\left(\frac{\text{Im}(H)}{\text{Re}(H)}\right) = -\text{atan}\left(\frac{\omega}{\omega_0}\right)$$

(rad or degree)

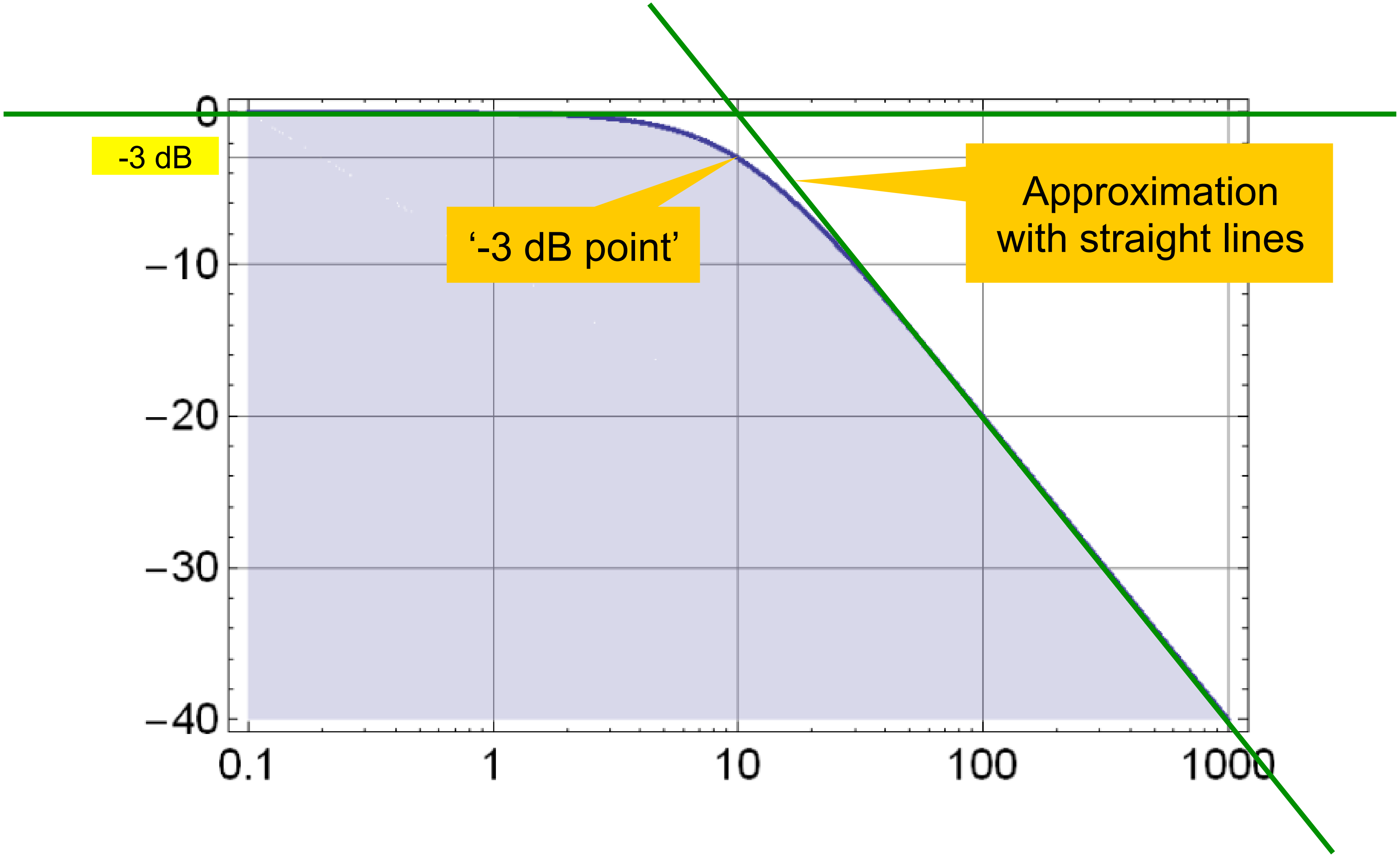
Bode Plot of LowPass (Amplitude)

VERY important!

- $\omega_0 = 10$

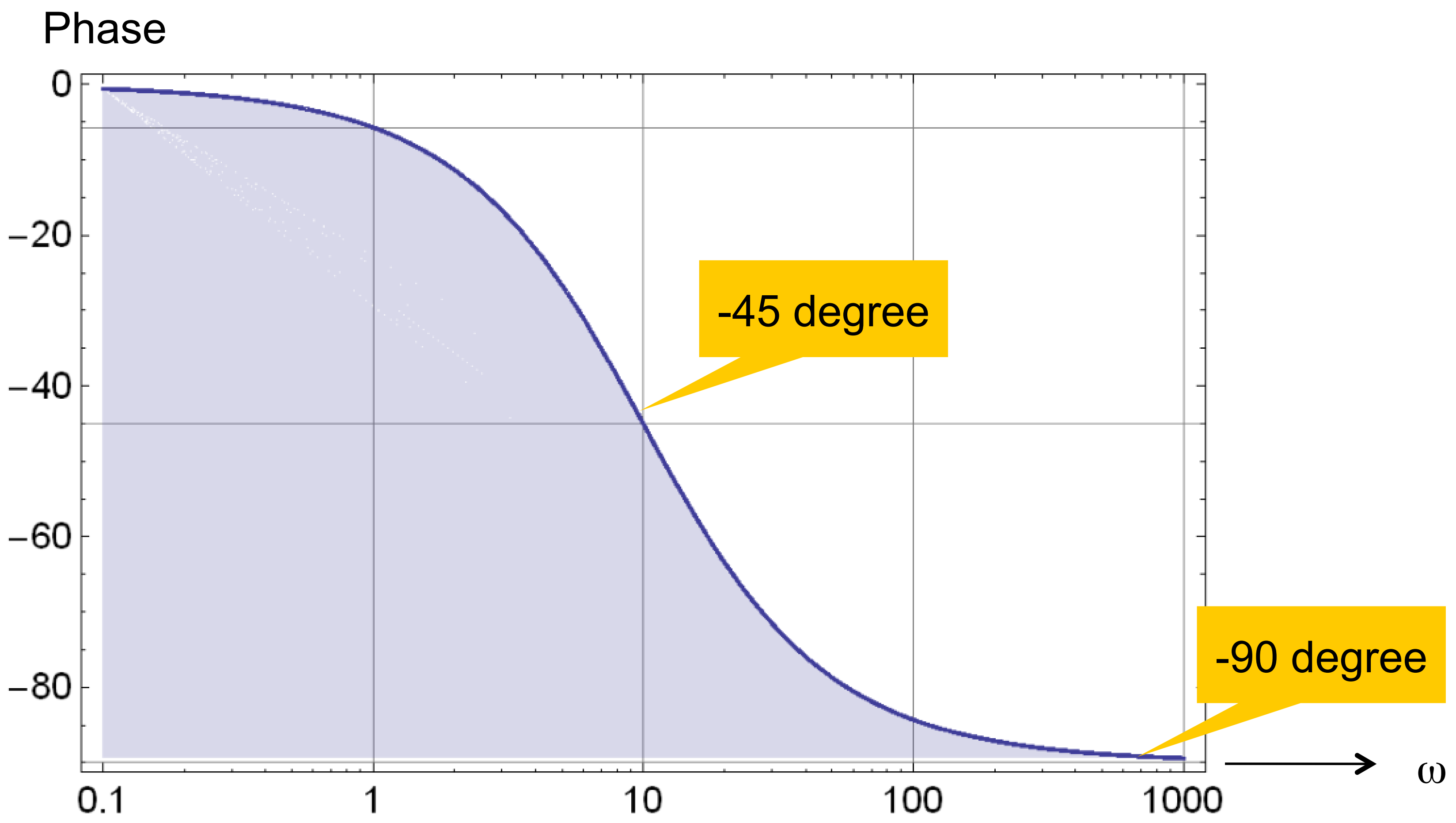


The same in dB



Bode Plot of LowPass (Phase)

- $\omega_0 = 10$
- Lin-Log Plot!



Where is the Corner?

$$H(\omega) = \frac{1}{1 + i\frac{\omega}{\omega_0}}$$

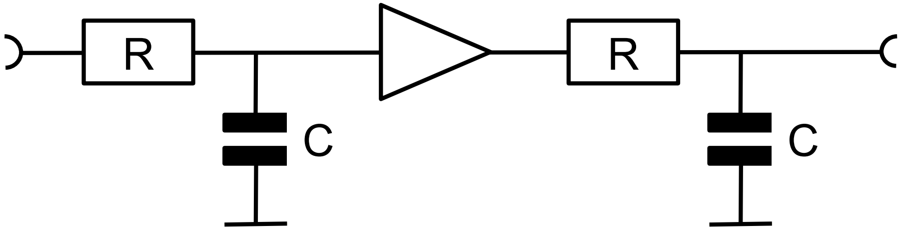
- At the corner frequency ω_0 = 1/(RC):
- The impedance of the capacitor is

  1/(sC) = 1/(i ω_0 C) = R/I

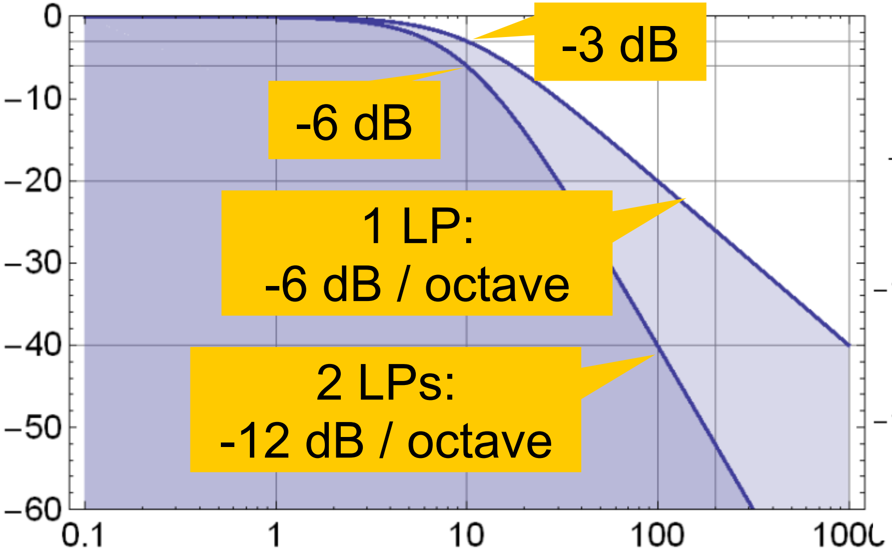
  with absolute value R.
- Therefore: At the corner frequency, the (absolute value) of the impedances of the capacitor and the resistor are the same.
  - C becomes 'more important' than R

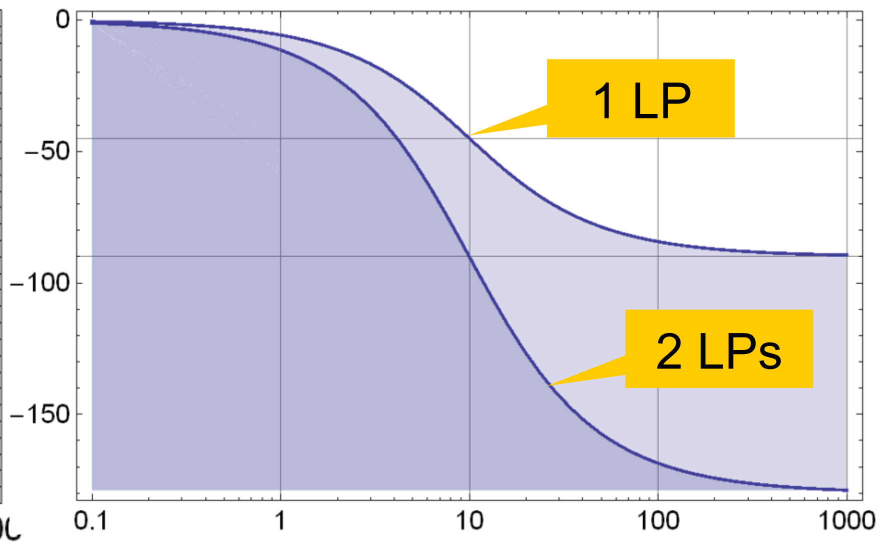
Series Connection of two Low Pass Filters

- Consider two identical LP filters. A ‘unit gain buffer’ makes sure that the second LP does not load the first one:



- From the properties of the LogLog Plot, the TF of the 2nd order LP is just the sum of two 1st order LPs:



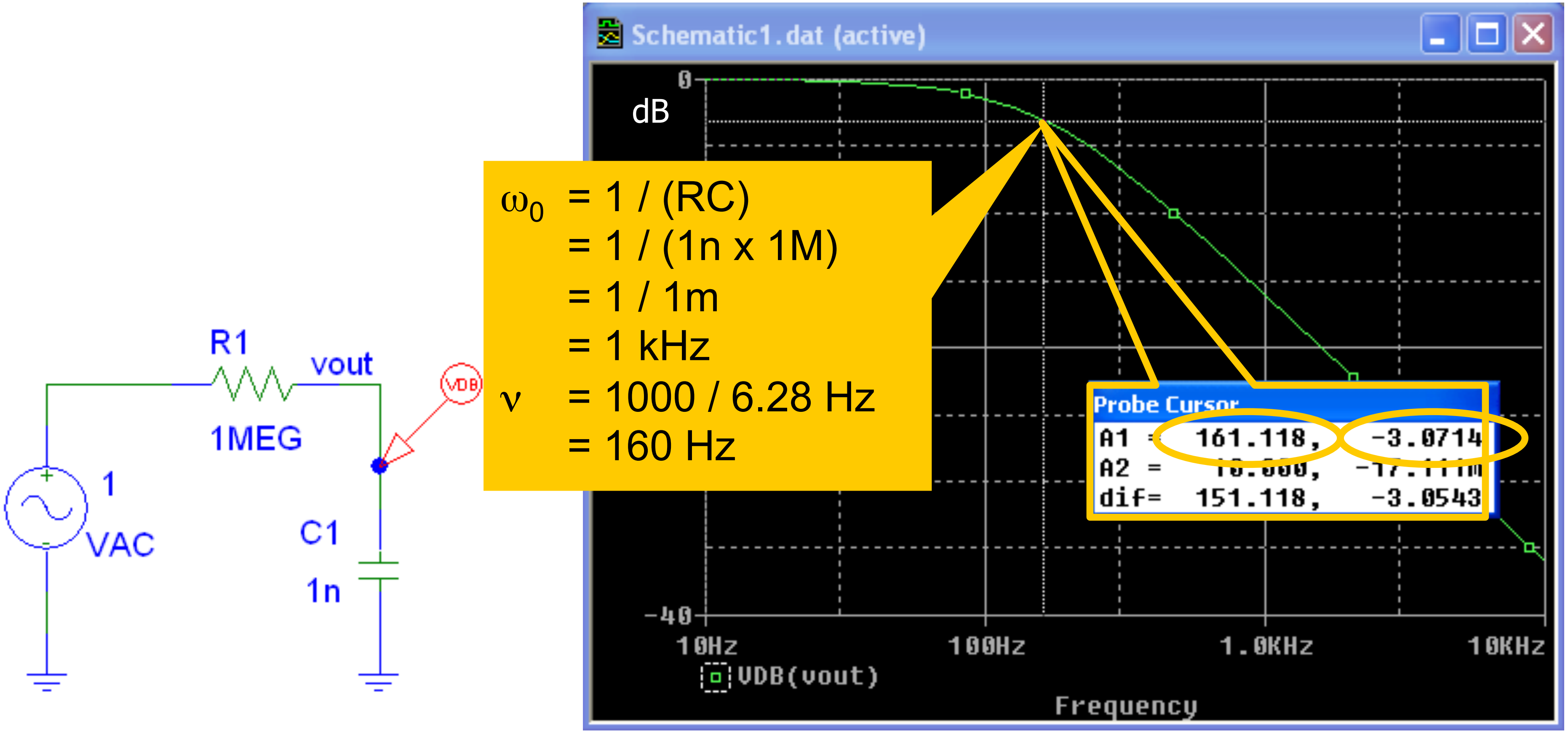


Why bother so much about the low pass ?

- All circuits behave like low-passes (at some frequency)!

Caveat!

- So far, frequency is expressed with ω, i.e. in radian / second
- We have: $\omega = 2\,\pi\,\nu$
- Therefore, the frequencies in Hertz are 2π lower!!!



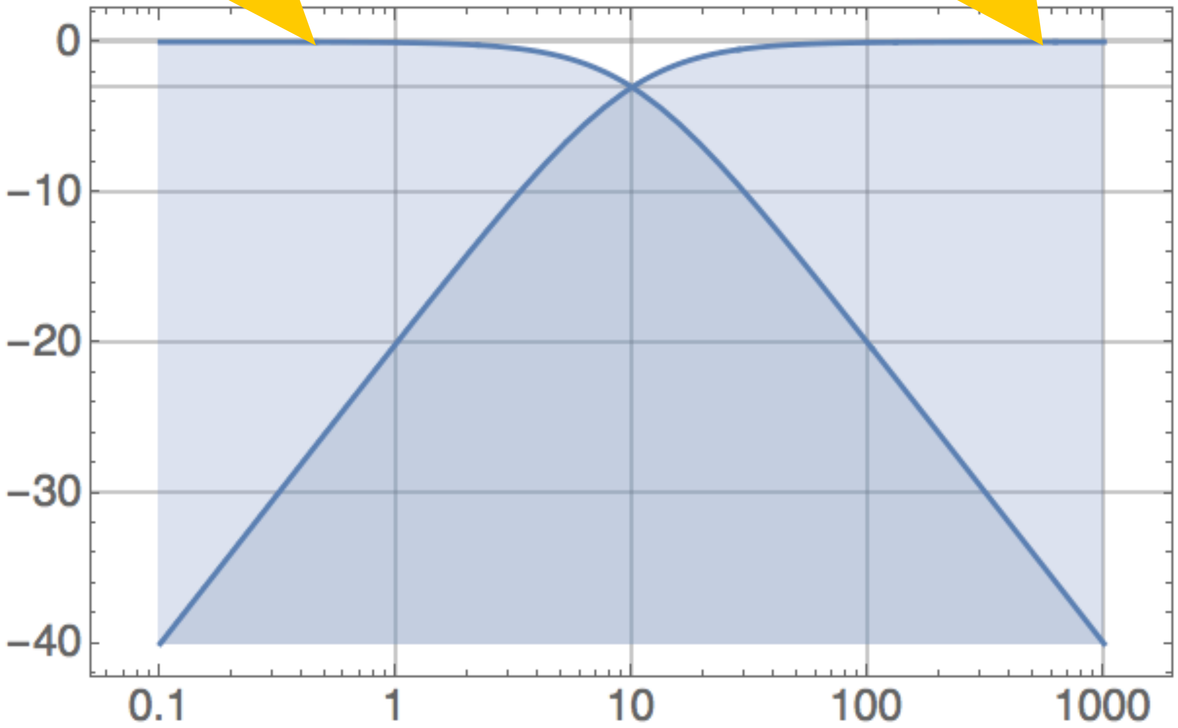
Low Pass and High Pass

$$LP[\omega] = \frac{1}{1 + i\frac{\omega}{\omega 0}}$$

$$HP[\omega] = \frac{i\frac{\omega}{\omega 0}}{1 + i\frac{\omega}{\omega 0}};$$

$$LPgain(\omega) = \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega 0}\right)^2}}$$

$$HPgain(\omega) = \frac{\frac{\omega}{\omega 0}}{\sqrt{1 + \left(\frac{\omega}{\omega 0}\right)^2}}$$



Bode Plots with Mathematica

- Replace s by i ω
- Calculate (squared) gain as absolute value

```
gain2 = H[i ω] Conjugate[H[i ω]] // ComplexExpand // Simplify
```

$$\frac{1 + 4 C^2 R^2 \omega^2}{1 + 7 C^2 R^2 \omega^2 + C^4 R^4 \omega^4}$$

- To plot, convert to dB by taking 20 $Log_{10}[\sqrt{H}]$.
  - The sqrt can be eliminated by using 10 $Log_{10}[H]$

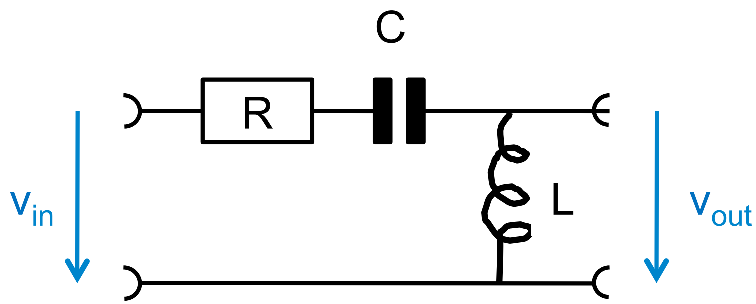
```
LogLinearPlot[10 Log[10, gain2] /. {R → 1, C → 1}, {ω, 0.01, 100}
 , PlotRange → {-20, 2}, Filling → -20]
```

- For phase, better use ArcTan[Re,Im] to get quadrant right

```
LogLinearPlot[180/π ArcTan[Re[H[i ω]], Im[H[i ω]]] /. {R → 1, C → 1}, {ω, 0.01, 100}]
```

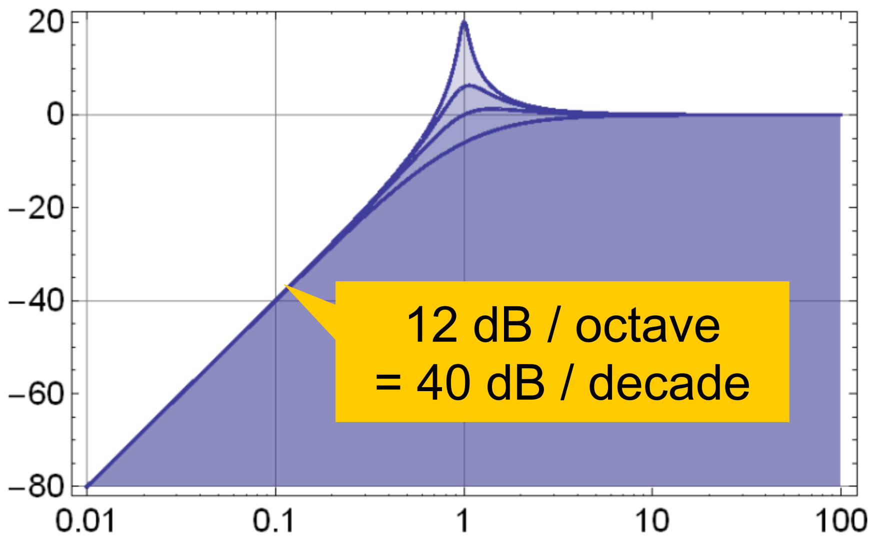
A More Complex Example

- Consider a (High Pass) filter with an inductor:



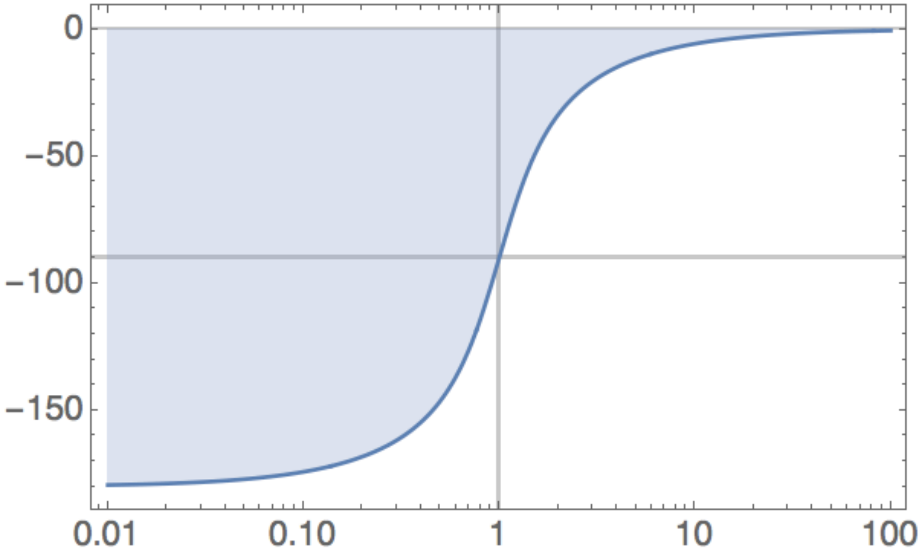
Mathematica Demo

- The transfer function is $H(s) = (C\,L\,s^2)/(1+C\,R\,s+C\,L\,s^2)$
- It is of 'second order' (s has exponent of 2 in denominator)
- Magnitude:
  L=C=1
  R=0.1,0.5,1,2
- 'Inductive peaking'

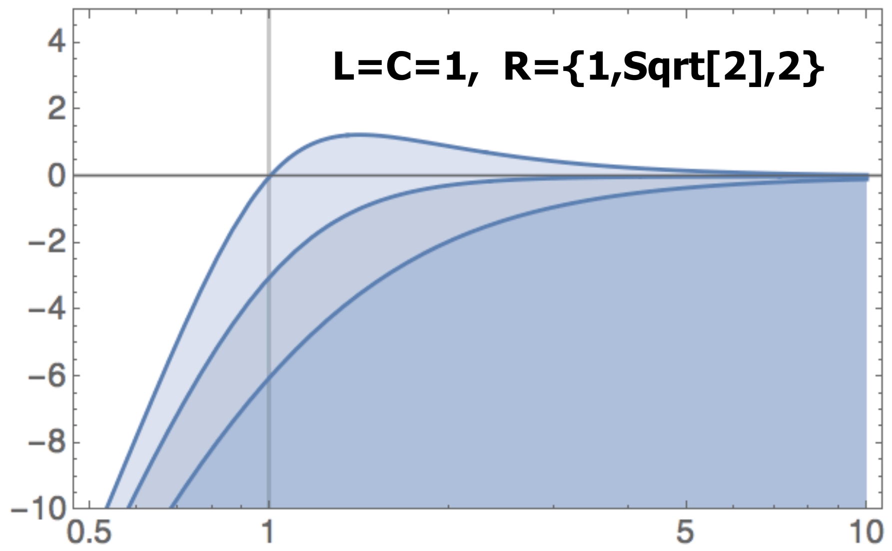


Phase

- Phase



- For fun:
  - When is filter steep & flat?
  - Zoom to corner frequency:



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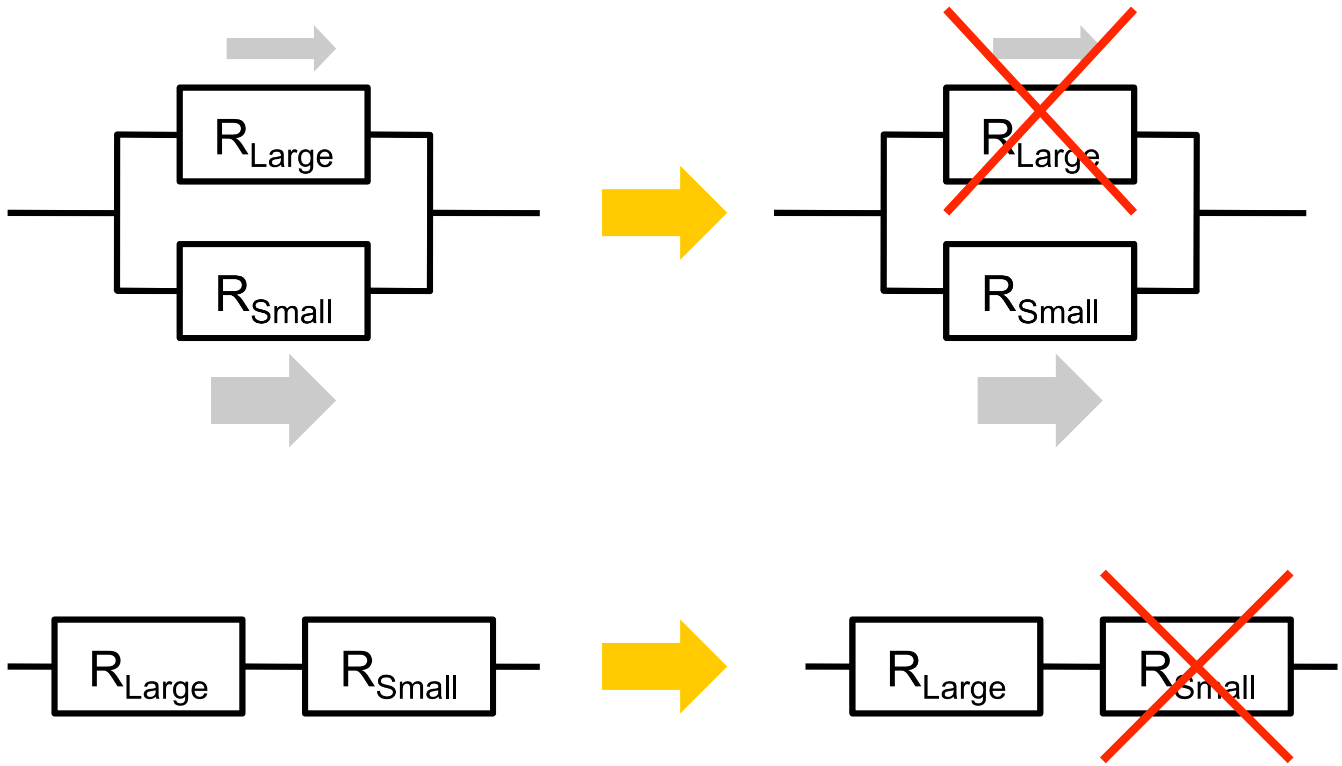
CIRCUIT SIMPLIFICATIONS

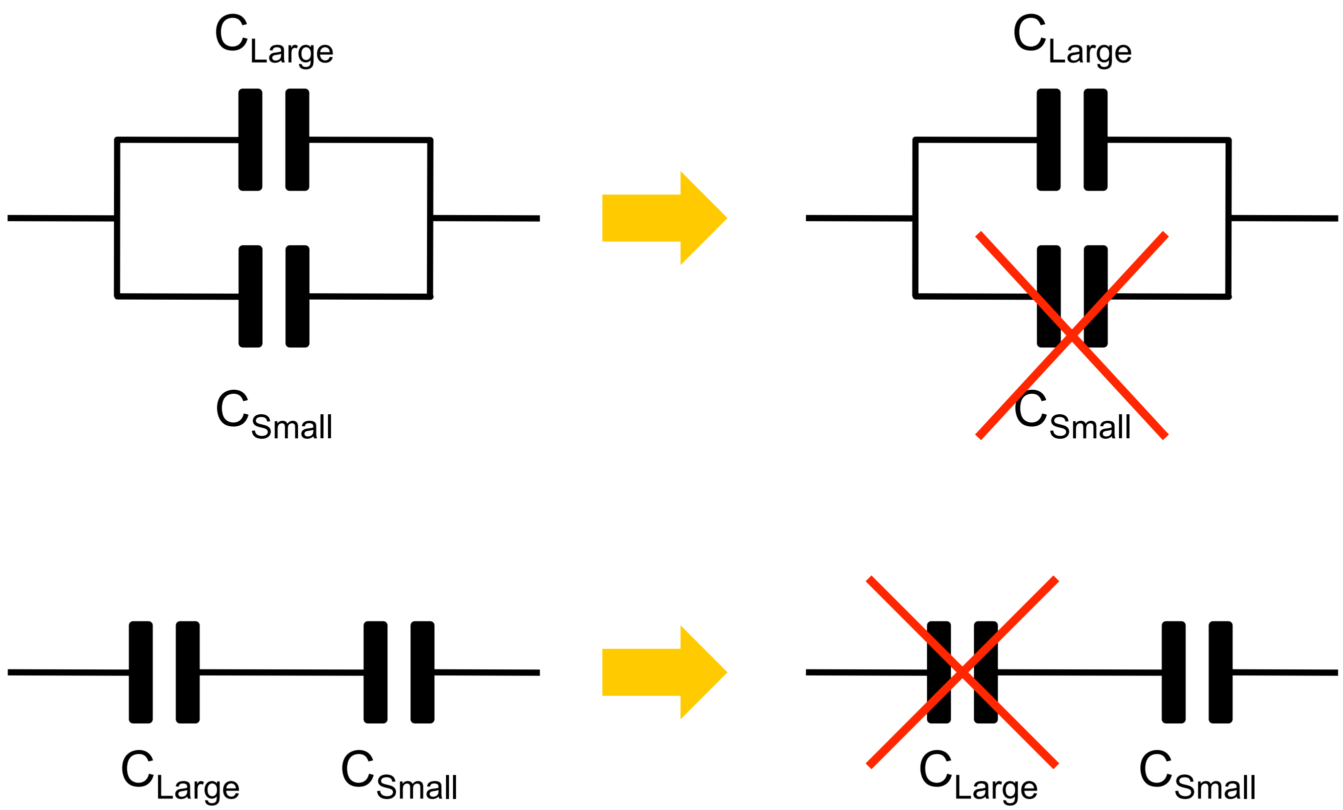
Large and Small Values

- To roughly understand behavior of circuits, only keep the dominant components:



- Eliminate *larger* or the *smaller* part (depending on circuit!)
- Error ~ ratio of components

The same for Capacitors



Resistors AND Capacitors

- Behavior depends on frequency ($|Z_C| = 1/(2\pi\nu C)$)