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TIME DOMAIN ANALYSIS
(A VERY SUPERFICIAL APPROACH)

Laplace Transform (LT)

- The Laplace Transform (LT) is an integral transform similar to the Fourier Transform
- The LT of a function f(t) is defined as

$$F(s) = \mathcal{L}\left\{f\right\}(s) = \int_0^\infty f(t)e^{-st}\,\mathrm{d}t,$$

  - s is a *complex* variable
  - This integral does not necessarily exist for all possible f(t) and s (If s has a real part >0, f(t) must not grow faster than C $e^{\mathrm{Re}(s)\,t}$)

- The Inverse Transform is more complicated:

$$\mathcal{L}^{-1}\left\{F(s)\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds$$

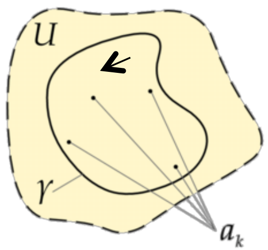
  - where γ > Re(all singularities of f).
  - This is a *line integral* in the *complex s-plane*, 'right' of all singularities

Reminder (hopefully..): Integration with Residues

This is very simplified!
The statements are valid under certain conditions only.
Consult a book on Complex Analysis!

- The *Residue Theorem* states that the line integral of a function *f(z)* along a *closed* curve γ in the complex z-plane is 2πi × (the sum of the residues at the *singularities* a_k of *f)*:

$$\oint_\gamma f(z)\,dz = 2\pi i \sum \operatorname{Res}(f, a_k)$$



Wikipedia

- The *residue* is a characteristic of a singularity a_k (or c below)
  - For a first order (simple) *pole* at c (where f behaves ~ like 1/z):

$$\operatorname{Res}(f, c) = \lim_{z \to c} (z - c) f(z).$$

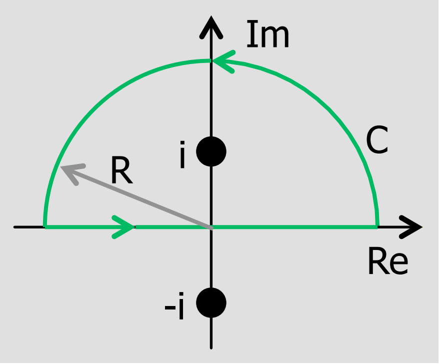
  - More generally, for a pole of order n:

$$\operatorname{Res}(f, c) = \frac{1}{(n-1)!} \lim_{z \to c} \frac{d^{n-1}}{dz^{n-1}} \left((z - c)^n f(z) \right).$$

Example for Integration with Residues

- Assume we want to find $A = \int_{-\infty}^{\infty} \frac{1}{x^2+1}$.
- The function $f(z) = \frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$ has poles i and $-i$
- The residue at the 'simple pole' i is:

$$Res(f,i) = \lim_{z \to i} f(z)(z-i) = \lim_{z \to i} \frac{1}{(z+i)} = \frac{1}{2i}$$



- The line integral along green curve C is

$$\int_C f(z)dz = 2\pi i \, Res(f,i) = \pi$$

- This is independent of R! With increasing R, the contribution of the upper arc vanishes (the length of the arc rises ~R, but f falls as $1/R^2$)
- Therefore $A = \int_{-\infty}^{\infty} \frac{1}{x^2+1} = \pi$

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Example 1: LT of 1

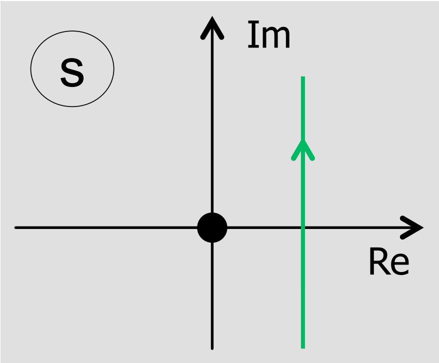
- For f(t) = 1:

$$L\{1\} = \int_0^\infty e^{-s\,t}\,dt = \left(\frac{-e^{-s\,t}}{s}\right)_0^\infty = \frac{-e^{-s\,\infty}}{s} + \frac{e^{-s\,0}}{s} = \frac{1}{s}$$

- Valid for Re[s] > 0 to make sure this vanishes

Example 1 (inverse): Inverse LT of 1/s

- For $F(s) = \frac{1}{s}$:



$$L^{-1}\left\{\frac{1}{s}\right\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\frac{1}{s}\,ds$$

- The integral has just one pole at s = 0.
- The Residuum is:

$$\mathrm{Res}\left[e^{st}\,\frac{1}{s},\,0\right] = \mathrm{Limit}\left[(s-0)\,e^{st}\,\frac{1}{s},\,s\to 0\right] = \mathrm{Limit}\left[e^{st},\,s\to 0\right] = 1$$

- So the Integral is

$$\int_{\gamma-i\infty}^{\gamma+i\infty}\ldots\,ds = 2\pi i\,\mathrm{Res}[\ldots,\,0] = 2\pi i$$

- And we just have $\boxed{L^{-1}\left\{\frac{1}{s}\right\} = 1}$

(the arc contribution at infinity vanishes again…)

Properties of Laplace Transforms

- For $f(t) = \mathcal{L}^{-1}\{F(s)\},\ g(t) = \mathcal{L}^{-1}\{G(s)\}$ we have:

Function	Laplace Transform	
$af(t) + bg(t)$	$aF(s) + bG(s)$	Linearity
$f'(t)$	$sF(s) - f(0)$	Derivative
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s) \cdot G(s)$	Convolution
$\int_0^t f(\tau)\,d\tau = (u * f)(t)$	$\frac{1}{s}F(s)$	Integration
$f(t-a)u(t-a)$	$e^{-as}F(s)$	Time Shift

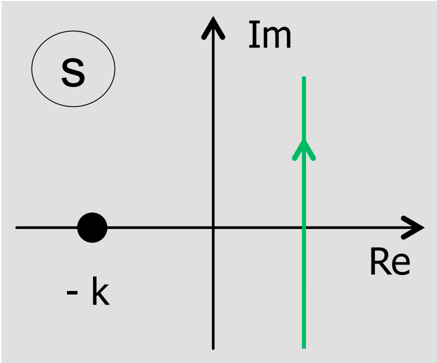
u(t): Step function

Example 2 ('frequency shift'):

- For $F(s) = \dfrac{1}{s+k}$:

$$L^{-1}\{F[s]\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\,\frac{1}{s+k}\,ds$$

- The pole is now at s = -k.



- The Residuum is:

$$\mathrm{Res}\left[e^{st}\,\frac{1}{s+k},\ 0\right] = \mathrm{Limit}\left[(s+k)\,e^{st}\,\frac{1}{s+k},\ s\to -k\right] = e^{-kt}$$

- And we just have $\boxed{L^{-1}\left\{\dfrac{1}{s+k}\right\} = e^{-kt}}$

Why is Laplace Transform so Useful ?

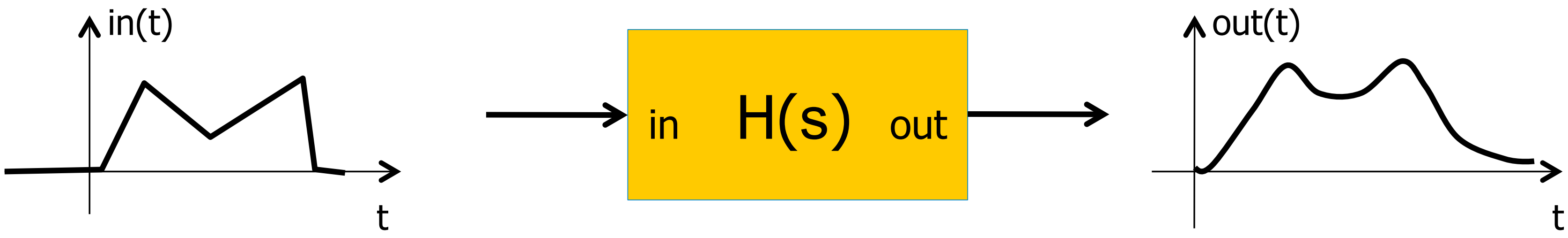
- Differential / Integral equations in t can be converted to Analytical equations in s, where they can be solved
- EQ(t) → transform to H(s) → Solve in s → Transform back

- Example: Radioactive Decay
  - f[t]: Number of atoms at time t
  - The # of decaying atoms is prop. to # of atoms: $\frac{df[t]}{dt} = -\lambda f[t]$
  - With F[s] = LT(f[t]): (f[0] = N_0 is initial number of atoms) $s F[s] - f[0] = -\lambda F[s]$, $f[0] \leftrightarrow N_0$
  - This can be solved in s-domain: $F[s] = \frac{N_0}{s + \lambda}$
  - Transforming back (see example) gives: $f[t] = N_0 e^{-\lambda t}$

Laplace Transform and Transfer Function

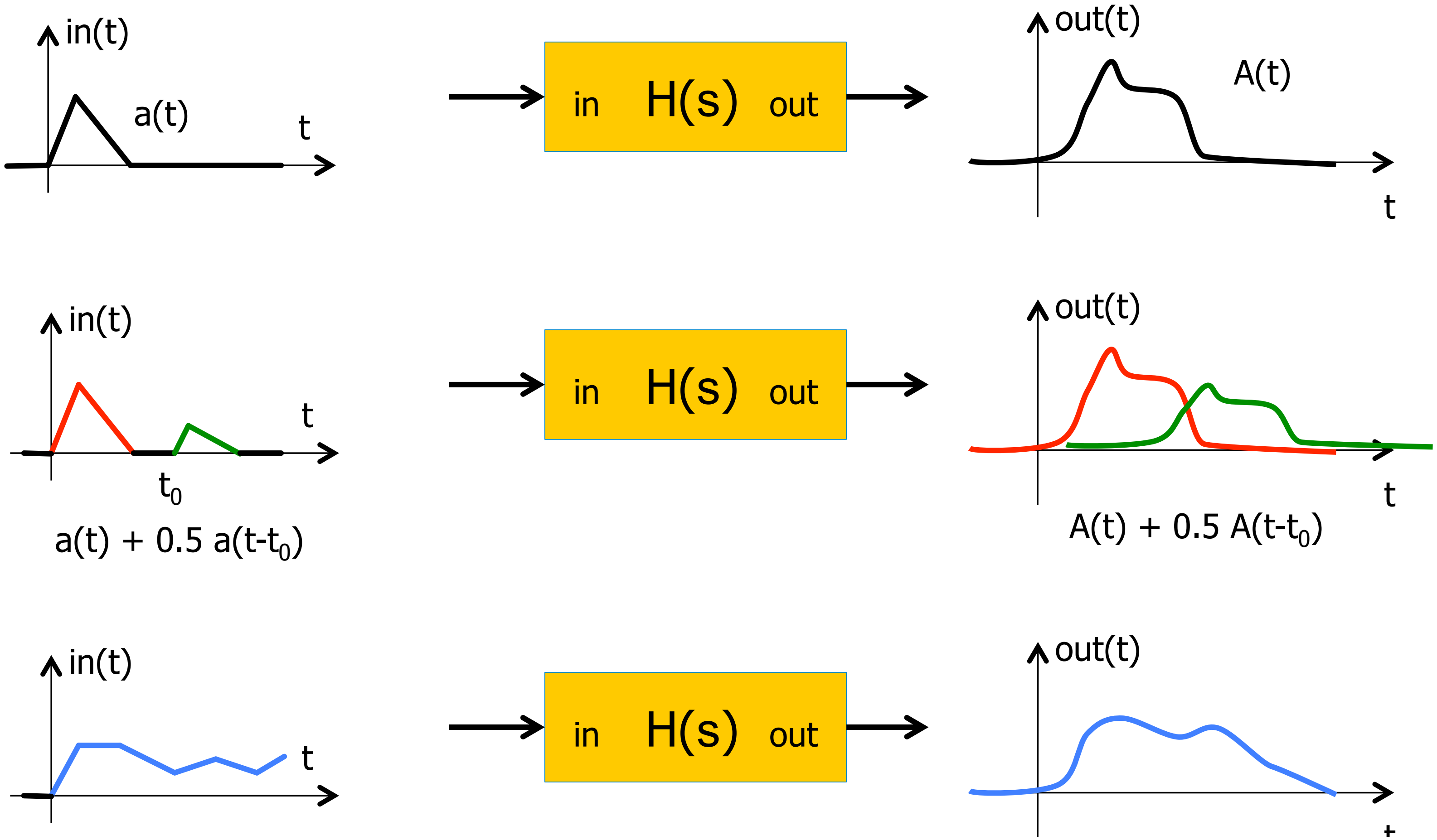
Time Response

- The **transfer function** tells us how sine inputs are modified by the system, i.e. what happen in the **frequency domain**
- How can be get the **time response** for an arbitrary input?



- For a *linear, time invariant (LTI)* system, we can use:
  - The response of a k × larger input pulse is just k × larger
  - The response for a time shifted input is time shifted
- For such a system we can
  - express the input signal as a superposition of 'simple' signals
  - Calculate the output for each 'simple' component
  - Superimpose the outputs

Illustration



$$\int_0^\infty k[\tau]\ a[t-\tau]\ d\tau$$

$$\int_0^\infty k[\tau]\ A[t-\tau]\ d\tau$$

Note that the integrals are CONVOLUTIONS (Faltung) of two functions!

Clever Choice of the 'nominal input' a[t]

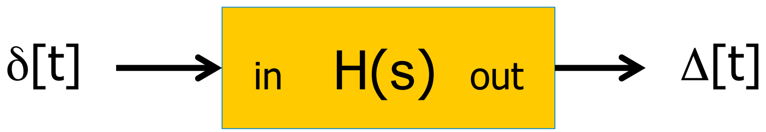
- To make the convolutions as simple as possible, it is best to chose a[t] to be Dirac Delta 'function'
- For any input function we can write

$$f_{in}[t] = \int_{-\infty}^{\infty} f_{in}[\tau]\ \delta[t-\tau]\ d\tau$$

- The output is then just

$$f_{out}[t] = \int_{-\infty}^{\infty} f_{in}[\tau]\ \Delta[t-\tau]\ d\tau$$

where Δ[t] is the response of the circuit to a δ[t] input, the ***'delta response***':



Note: I am a bit sloppy here with integration limits..

What is the Delta Response Δ[t] ?

- We do not know Δ[t], but: it turns out that its LT is just the transfer function!

> The Laplace Transform of the Delta Response of a circuit is just given by its transfer function H[s]

- Knowing that LT(Δ[t]) = H[s], what is Δ[t] ?
  It's the Inverse LT:

$$\Delta[t] = LT^{-1}\{H[s]\}$$

- Why is this?
  - If we write down Kirchhoff's rules in the time domain, we get differential / integral equations.
  - The 'topology' of the equations is the same as using complex impedances.
  - If we transform this, we can get the impulse response

General Time Response

- Start from $f_{out}[t] = \int_{-\infty}^{\infty} f_{in}[\tau]\ \Delta[t-\tau]\, d\tau$
- Laplace transform both sides and use Convolution rule:

$$F_{out}[s] = LT\left\{\int_{-\infty}^{\infty} f_{in}[\tau]\ \Delta[t-\tau]\, d\tau\right\} = F_{in}[s]\ LT\{\Delta[t]\}$$

- Use our knowledge that $LT\{\Delta[t]\} = H[s]$

$$F_{out}[s] = F_{in}[s]\, H[s]$$

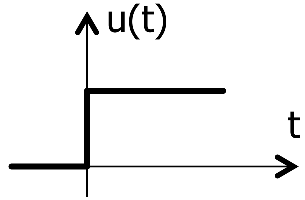
- And transform back:

$$\boxed{f_{out}[t] = LT^{-1}\{\, LT\{f_{in}[t]\}\ H[s]\,\}}$$

To calculate the time response of a circuit to an **arbitrary** input f[t]:
1. Laplace Transform f[t], yielding F[s]
2. Multiply with the Transfer function H[s]
3. Laplace Transform back

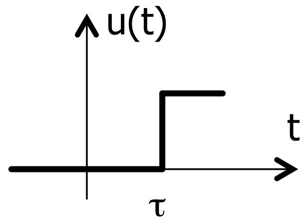
Important Input Functions

- The most important input to test a circuit is the Unit step:
  - It is often called u[t], Heaviside Step function, UnitStep,…



$$\xrightarrow{\text{LT}} \quad \frac{1}{s}$$

- For a Shifted Step, use Time Shift rule:



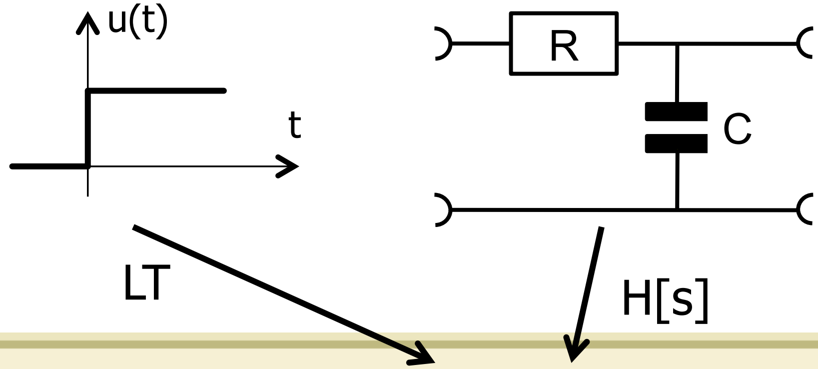
$$\xrightarrow{\text{LT}} \quad \frac{e^{-s\tau}}{s}$$

- A rectangular Pulse is just the difference of two Unit Steps

- For very short input signals (charge deposition in detector), input is the Dirac Delta, with LT = 1.

Example 1: Step Response of Low Pass

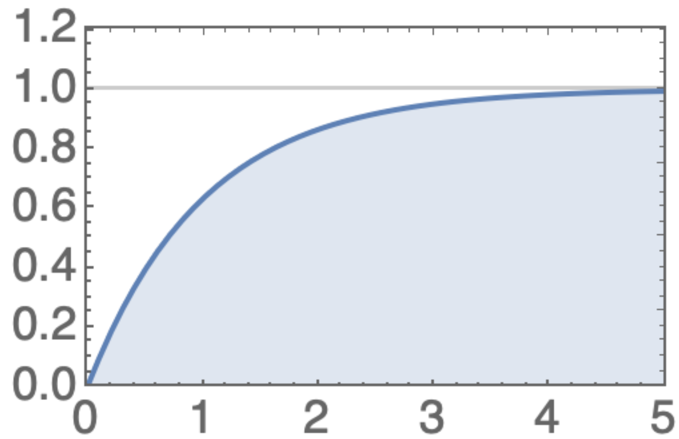
- Consider




```
InverseLaplaceTransform[1/s 1/(1 + s τ), s, t]
```

$$\text{InverseLaplaceTransform}\left[\frac{1}{s}\,\frac{1}{1+s\,\tau},\ s,\ t\right]$$

$$1 - e^{-\frac{t}{\tau}}$$



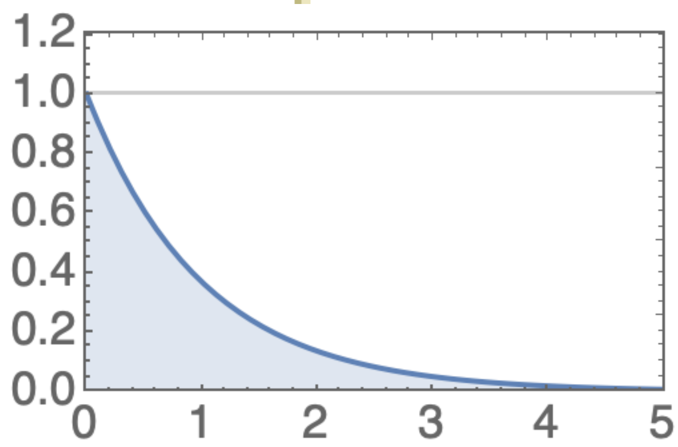
- This is the ***step response***

Watch out!

- What is $LT^{-1}[H[s]]$???, e.g.

```
InverseLaplaceTransform[1/(1 + s τ), s, t]
```

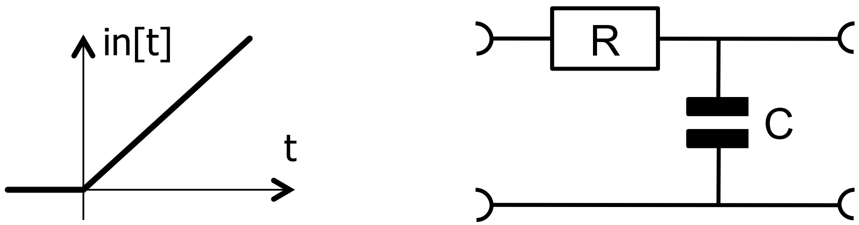
$$\frac{e^{-\frac{t}{\tau}}}{\tau}$$



- That is the output for a Dirac pulse at the input (with LT[δ[t]] = 1), i.e. that is the ***impulse response***

Example 2: Response of Low Pass to Slope

- Now Consider a linear input ramp in[t] = k t




```
In[301]:= IN[s_] = LaplaceTransform[k t UnitStep[t], t, s]
```

- The LT is

Out[301]= $\frac{k}{s^2}$

- So our response is

```
f[t_] = InverseLaplaceTransform[IN[s] H[s], s, t]
```

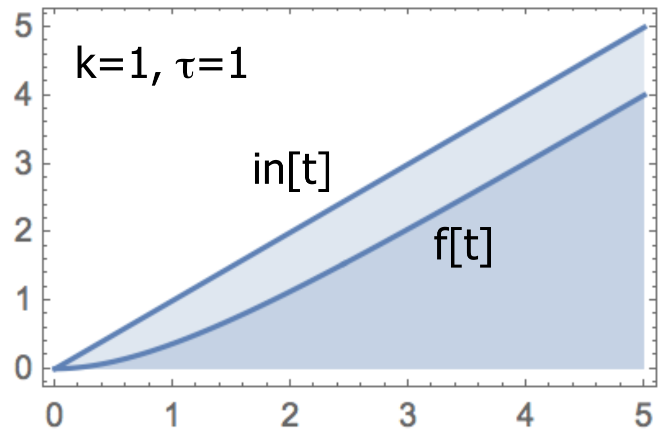
$$k\left(t - \tau + e^{-\frac{t}{\tau}}\,\tau\right)$$

```
In[46]:= Limit[D[f[t], t], t → ∞]
```

Out[46]= k

```
In[47]:= Limit[f[t] - k t, t → ∞]
```

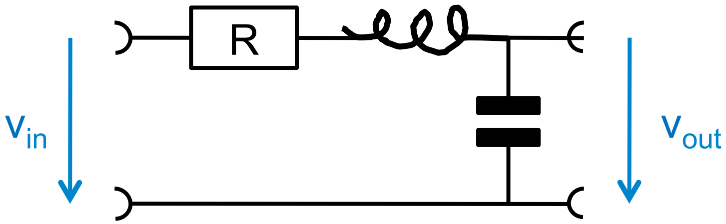
Out[47]= $-k\,\tau$



Example 3: Step Response of RLC

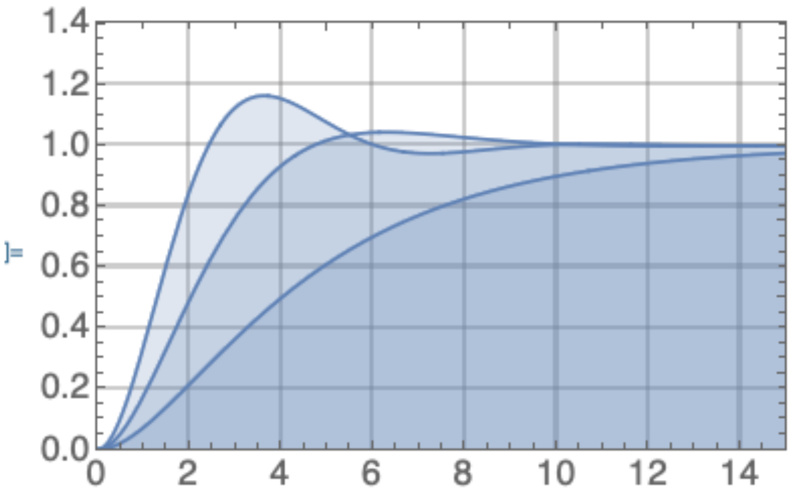
- Consider the RLC circuit

$$H[s] = \frac{1}{1 + C\,R\,s + C\,L\,s^2}$$




```
f[t_] = InverseLaplaceTransform[HH/s, s, t] // FullSimplify
```

$$1 - e^{-\frac{R\,t}{2\,L}}\,\mathrm{Cosh}\left[\frac{\sqrt{-\frac{4\,L}{C} + R^2}\;t}{2\,L}\right] - \frac{e^{-\frac{R\,t}{2\,L}}\,R\,\mathrm{Sinh}\left[\frac{\sqrt{-\frac{4\,L}{C}+R^2}\;t}{2\,L}\right]}{\sqrt{-\frac{4\,L}{C} + R^2}}$$



R=1, L=1, C=1,2,5