Exercise: Transfer Functions, Filters

Prof. Dr. P. Fischer

Lehrstuhl für Schaltungstechnik und Simulation
Uni Heidelberg

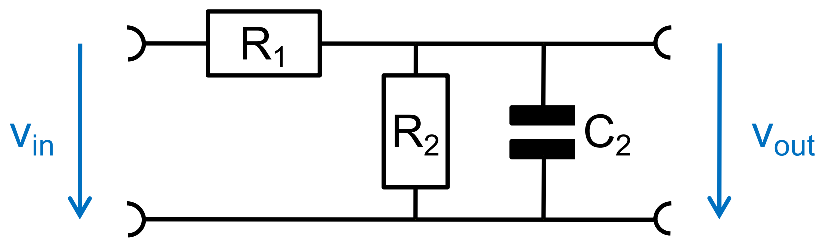
Recommendations

- I strongly recommend to use a mathematical program (Mathematica, Maple, SageMath,..) to solve the exercises

- For transfer functions, inspect each result:
  - What happens for $\omega \rightarrow 0,\infty$?
  - What happens if component values go to 0 or ∞?

Exercise 1

- Derive the Transfer Function of this circuit:



- Use 3 different approaches:
  - Treat the circuit directly (using Kirchhoff's rule)
  - Consider it as a voltage divider of two Impedances. Use R_1 for Z_1 and the parallel connection of R_2 and C_2 for Z_2
  - Replace the (resistive) voltage divider (R_1,R_2) by its Thévenin equivalent and then add the capacitor

- Make a Bode Plot
  - Observe the difference to the normal Low Pass Filter

Solution 1

Direct Treatment:

```
EQ = (Vin - Vout)/R1 == Vout s C2 + Vout/R2;
```

$$EQ = \frac{Vin - Vout}{R1} == Vout\ s\ C2 + \frac{Vout}{R2};$$

```
Solve[EQ, Vout] // First
```

$$\left\{Vout \to \frac{R2\ Vin}{R1 + R2 + C2\ R1\ R2\ s}\right\}$$

$$Hdirect = \frac{Vout}{Vin}\ /.\ \%$$

$$\frac{R2}{R1 + R2 + C2\ R1\ R2\ s}$$

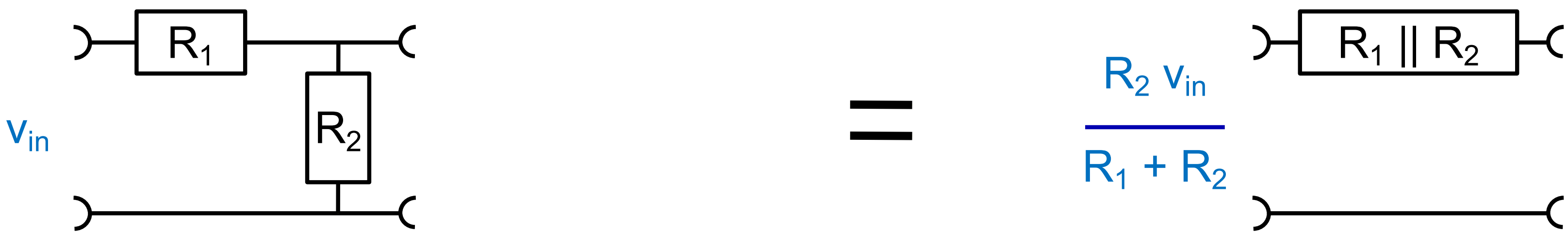
Voltage Divider:

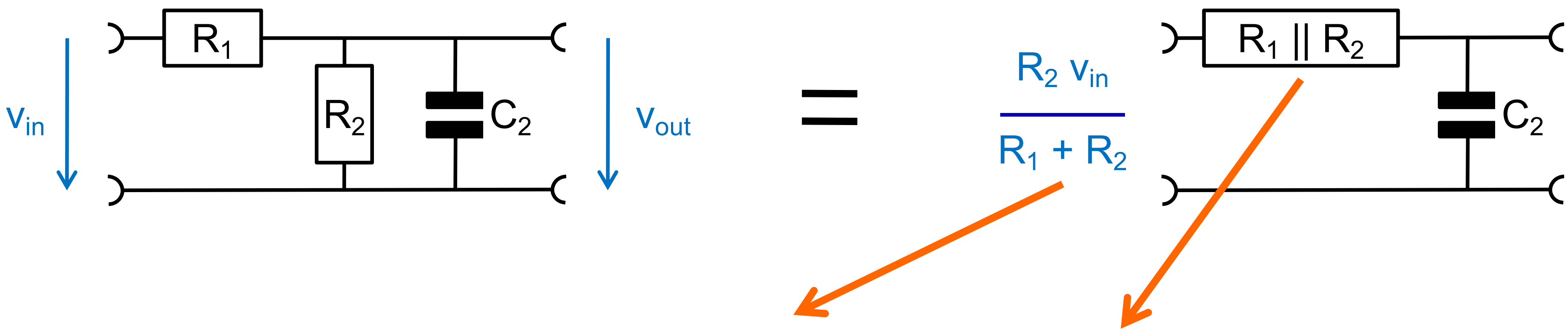
$$Hdiv = \frac{Z2}{Z1 + Z2}\ /.\ \left\{Z1 \to R1,\ Z2 \to \left(\frac{1}{R2} + s\ C2\right)^{-1}\right\}\ //\ Simplify$$

$$\frac{R2}{R1 + R2 + C2\ R1\ R2\ s}$$

```
Hdirect == Hdiv
True
```

Solution 1: Thévenin





$$\text{Hthenevin} = \frac{g}{1 + s\,RR\,C2} \;/.\; \left\{ g \to \frac{R2}{R1 + R2},\; RR \to \left(\frac{1}{R1} + \frac{1}{R2}\right)^{-1} \right\} \;//\; \text{Simplify}$$

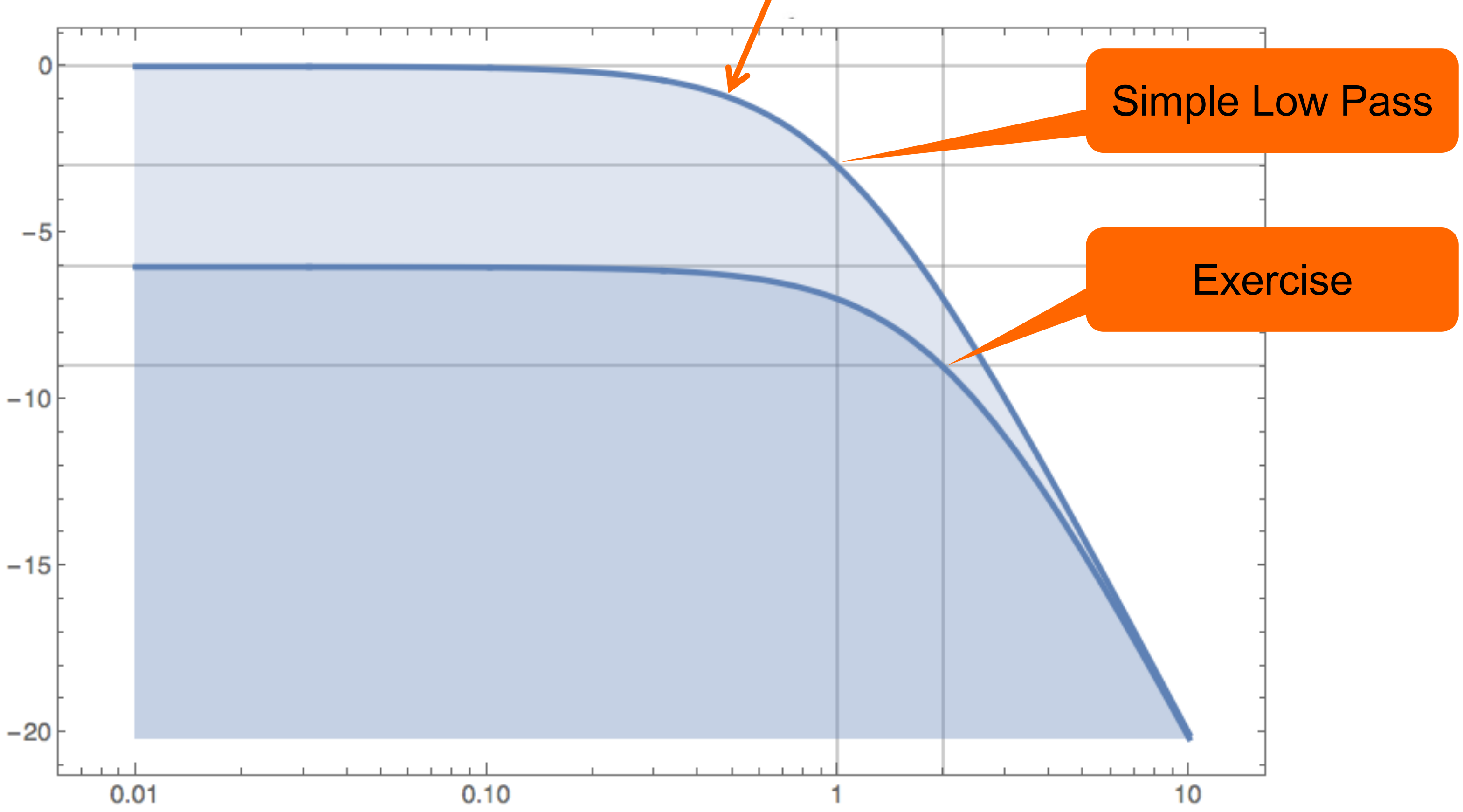
$$\frac{R2}{R1 + R2 + C2\,R1\,R2\,s}$$

Simple Low Pass

Solution 1

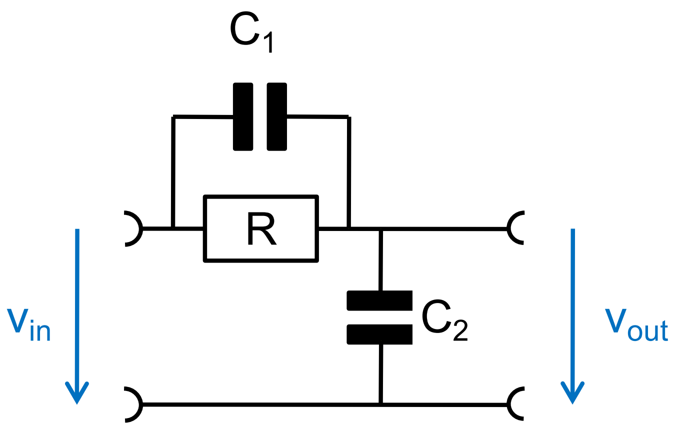
- Compared to the 'simple' Low-Pass:
  - The signal is attenuated by $R_1/(R_1+R_2)$
  - The time constant is lowered (i.e. the corner frequency is raised)
- Plot for $R_1 = R_2 = C_2 = 1$:

```
HLP = Sqrt[1/(1 + i ω RC) Conjugate[1/(1 + i ω RC)]] /. {RC → 1}
```



Exercise 2

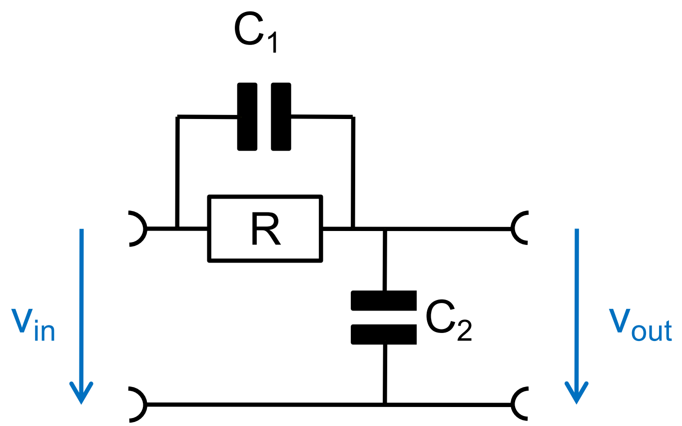
- Analyze the following circuit (simulation & calculation!):



  - What is the transfer function ?
  - At which frequencies are the ‘pole’ in the denominator and the ‘zero’ in the nominator ?
  - What are gain and phase for $s \rightarrow 0$ and for $s \rightarrow \infty$? Why?
  - What happens for $C_1 \rightarrow 0$, for $R \rightarrow 0$, for $R \rightarrow \infty$? Reasonable?
  - Simulate the circuit for $C_1 = C_2 = 10\text{pF}$ and $R = 10\ \text{k}\Omega$. Plot gain and phase!
  - Chose values so that the circuit attenuates to 1/10 at high frequencies.
  - For fun: At which frequency is phase shift maximal?

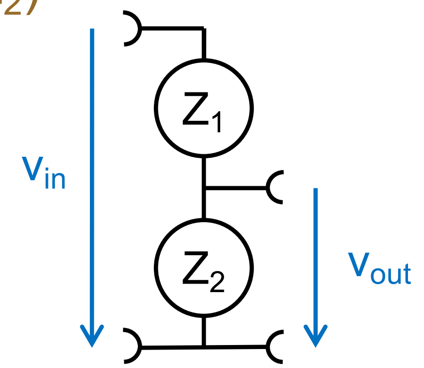
Solution 2

- Two possibilities:
  1. Treat circuit as voltage divider with C_1 // R and C_2
  2. Use Kirchhoff's law @ node v_{out}



- Voltage divider:
  - For any Z_1, Z_2, we have $v = v_{out}/v_{in} = Z_2 / (Z_1+Z_2)$
  - With $1/Z_1 = 1/R + s\,C_1$ and $1/Z_2 = s\,C_2$:

$$v = \frac{1 + C_1 R s}{1 + (C_1 + C_2) R s}$$



- Kirchhoff:
  - Solve $(v_{in}-v_{out})/R+(v_{in}-v_{out})\,sC_1 = v_{out}\,s\,C_2$ for v_{out}

Solution 2

- Limits
  - s → 0: caps are gone. v is just 1. No phase shift.
  - s → ∞: R can be neglected. frequency dependencies cancel. This is just a capacitive voltage divider. No phase shift

$$v = \frac{1 + C_1 R s}{1 + (C_1 + C_2) R s}$$

- Phase shift:

```
gain = Sqrt[(HH /. s → ⅈ ω) (HH /. s → -ⅈ ω)] // FullSimplify
```

$$\sqrt{\frac{1 + C1^2 R^2 \omega^2}{1 + (C1 + C2)^2 R^2 \omega^2}}$$

```
phase = -180/π ArcTan[Im[ComplexExpand[HH /. s → ⅈ ω]] / Re[ComplexExpand[HH /. s → ⅈ ω]]] // FullSimplify
```

$$\frac{180 \, \mathrm{ArcTan}\left[\frac{C2\, R\, \omega}{1 + C1\,(C1 + C2)\, R^2 \omega^2}\right]}{\pi}$$

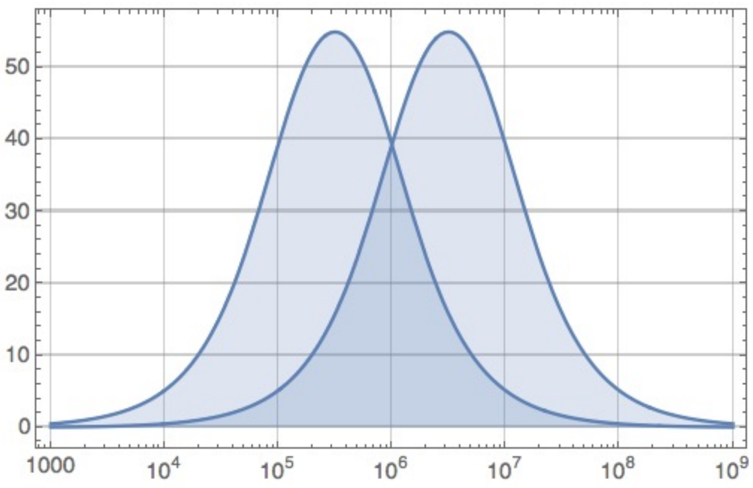
```
PhaseDeriv = D[phase, ω] // FullSimplify
```

$$\frac{180\, C2\, R \left(1 - C1\,(C1 + C2)\, R^2 \omega^2\right)}{\pi + \left(2\, C1^2 + 2\, C1\, C2 + C2^2\right) \pi R^2 \omega^2 + C1^2 (C1 + C2)^2 \pi R^4 \omega^4}$$

```
ωmax = ω /. Solve[PhaseDeriv == 0, ω] // Last // FullSimplify
```

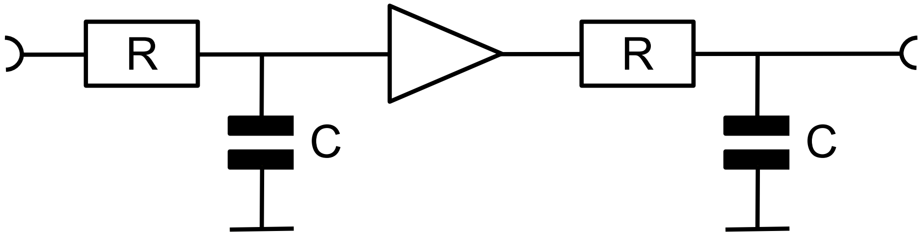
$$\frac{1}{\sqrt{C1\,(C1 + C2)\, R^2}}$$

```
LogLinearPlot[phase /. PARAM, {ω, 1×10^3, 1×10^9}]
```



Exercise 3: Cascaded Stages

- Consider the following two stage circuit (again):



- The triangle is a (voltage) buffer with infinite input impedance (it does not load the first low-pass) and zero output impedance. For simulation use a vcvs (voltage controlled voltage source) from analogLib with gain 1
- What transfer function do you expect ?
- Simulate the circuit !
- **Simulate** a version **without** buffer in the same schematic
- Where are differences ?
- Use a much larger R and correspondingly smaller C in the second low pass.
- Now **calculate** the exact transfer function **without** buffer

Solution 3

- Transfer Function with buffers

```
HH1 [s_] = (1/(1 + s R C))^2; (* with buffers: square of s

gain1[ω_] = Sqrt[HH1[ⅈ ω] HH1[-ⅈ ω]] // FullSimplify
```

$$\frac{1}{1 + C^2 R^2 \omega^2}$$

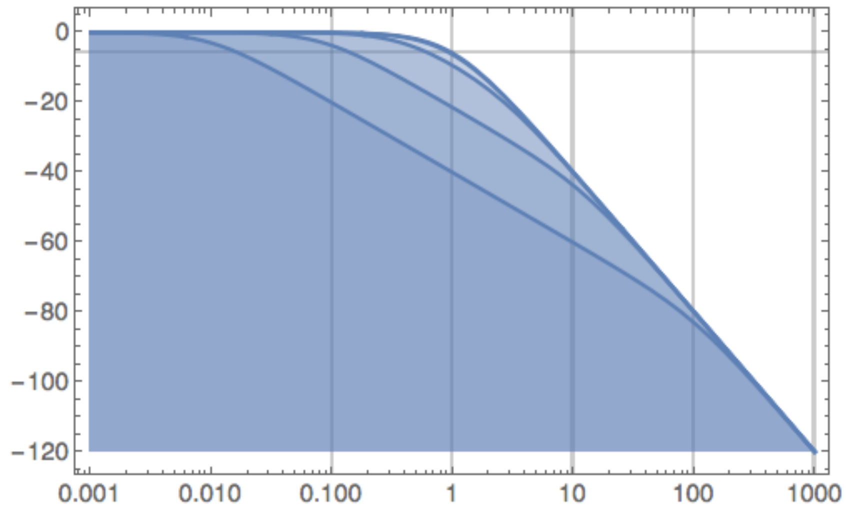
- Without Buffers:

```
EQ1 = (vin - v1)/R1 == v1 s C1 + (v1 - vout)/R2; (* node v1 *)
EQ2 = (v1 - vout)/R2 == vout s C2; (* output node *)

Eliminate[{EQ1, EQ2}, v1] // Simplify

vin == (1 + C2 (R1 + R2) s + C1 R1 s (1 + C2 R2 s)) vout

HH2 [s_] = vout/vin /. Solve[%, vout] // First
```

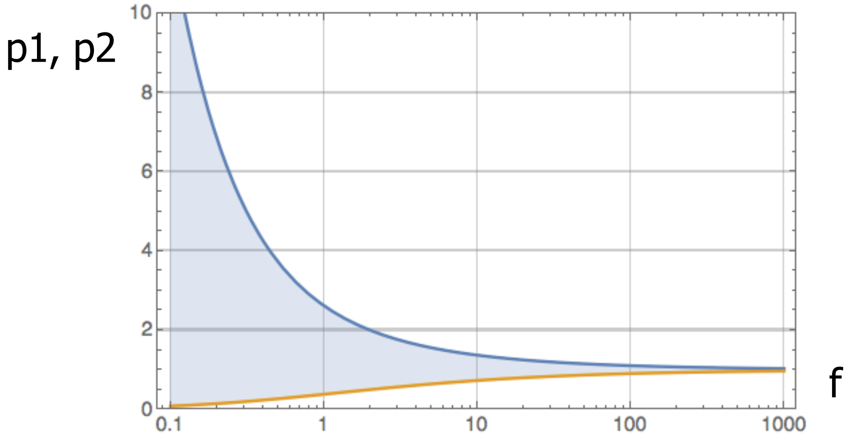
$$\frac{1}{1 + C1\,R1\,s + C2\,R1\,s + C2\,R2\,s + C1\,C2\,R1\,R2\,s^2}$$

Solution 3

- Bode Plot for different RC combinations in second stage:
  - We note two poles.
  - They coincide, when the second low pass does not load the first

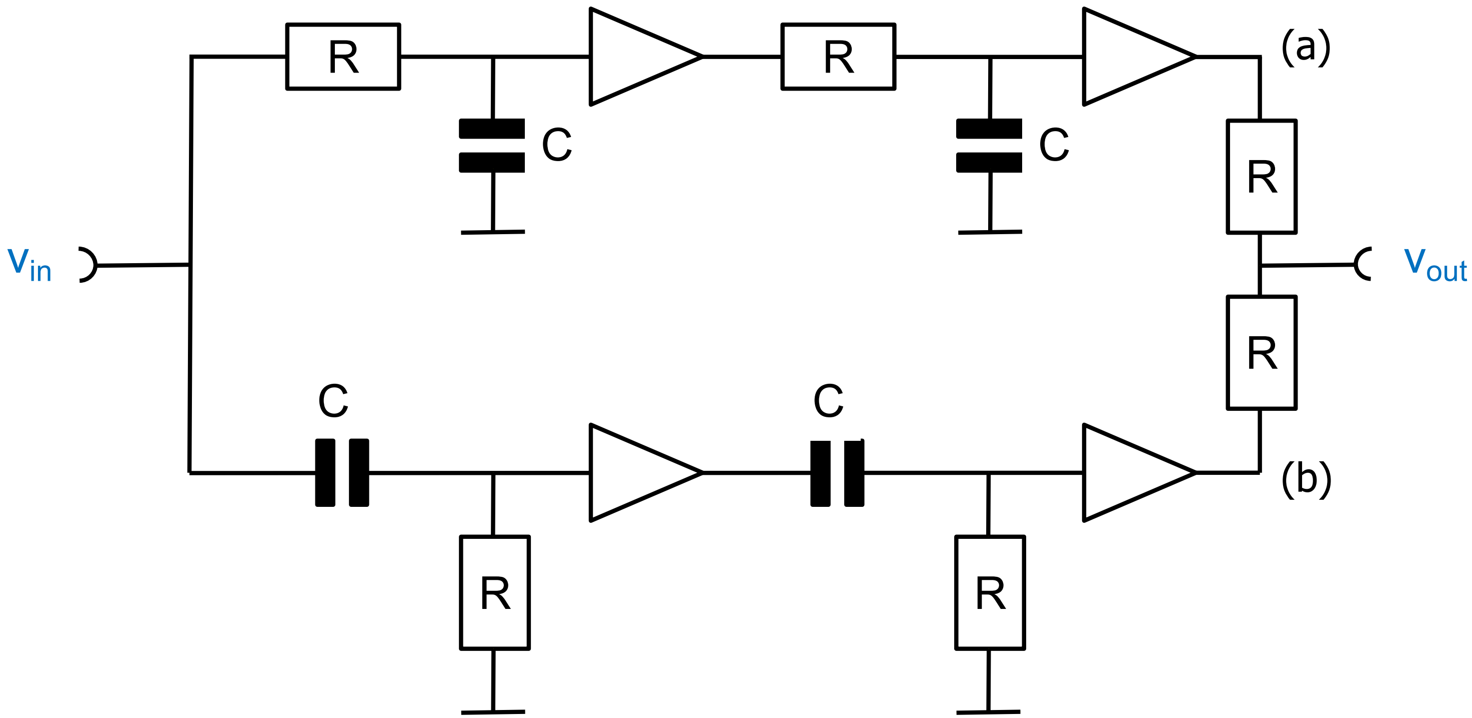


- Plot the poles as a function of *f* where $R_2 = R_1 f$, $C_2 = C_1 / f$, so that $R_2C_2 = R_1C_1$:



Exercise 4: Notch Filter

- Consider the following circuit made of cascaded High- and Low Pass stages:
  - The resistors at the output just add the signals at (a) and (b)



- What is the output signal at the corner frequency?
  - Explain this by comparing amplitudes ***and phases*** at (a) and (b)

Solution 4

```
$Assumptions = ω > 0 && RC > 0; HLP = 1/(1 + i ω RC); HHP = (i ω RC)/(1 + i ω RC);

vout = Vb + 1/2 (Va - Vb) // Simplify (* Output is averge of Va and Vb *)
```

$\frac{Va + Vb}{2}$

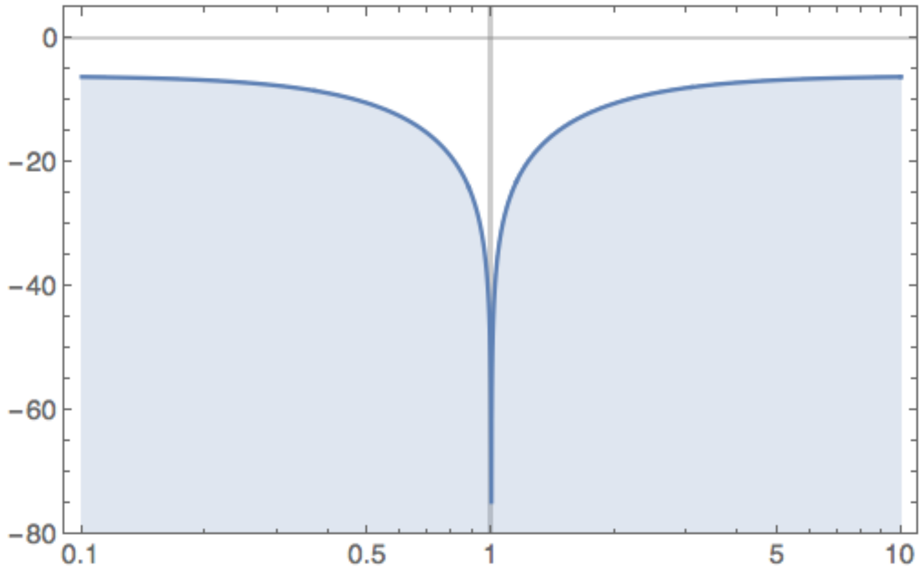
```
H = (HLP HLP + HHP HHP)/2 // Simplify
```

$\frac{-1 + RC^2\,\omega^2}{2\,(-i + RC\,\omega)^2}$

```
HMag = H Conjugate[H] /. RC → 1 // FullSimplify
```

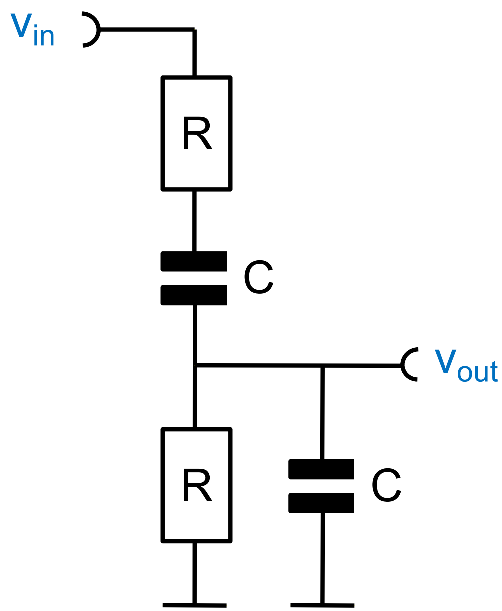
$\frac{(-1 + \omega^2)^2}{4\,(1 + \omega^2)^2}$

```
LogLinearPlot[dB[√HMag], {ω, 0.1, 10}
 , GridLines → {{1}, {0}}, PlotRange → {-80, 5}, Filling → -80]
```



- At the corner frequency, the signal is fully stopped!
- This is because **the phases** of the two signals are ± 90°, i.e. the signals are complementary
  - (A bit tricky to verify in Mathematic due to jump in ArcTan[]..)

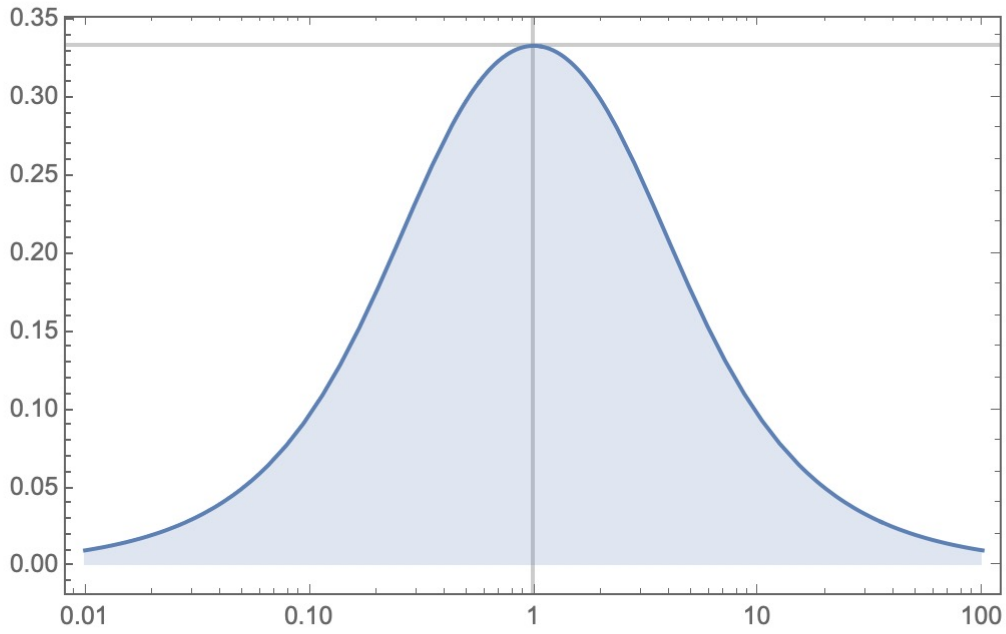
Exercise 5: Wien Bridge / Oscillator



- Consider this circuit:
- What is the transfer function?
- What is the magnitude at the center frequency?
- What is the Phase at the center frequency?
- Simulate the circuit for R=1k C=1n
- You can use this 'Wien Bridge' to make an oscillator:
  - Amplify v_{out} by *exactly* 3 (vcvs !) and feed the signal back to v_{in}.
  - Set an initial condition of 1V (parameter!) for the lower C and start a transient simulation.
  - How does this work?
  - What happens if the gain is not exactly 3 ?

Solution 5

- $H[s] = \dfrac{C\,R\,s}{1 + 3\,C\,R\,s + C^2\,R^2\,s^2}$, gain = $\dfrac{C\,R\,\omega}{\sqrt{1 + 7\,C^2\,R^2\,\omega^2 + C^4\,R^4\,\omega^4}}$



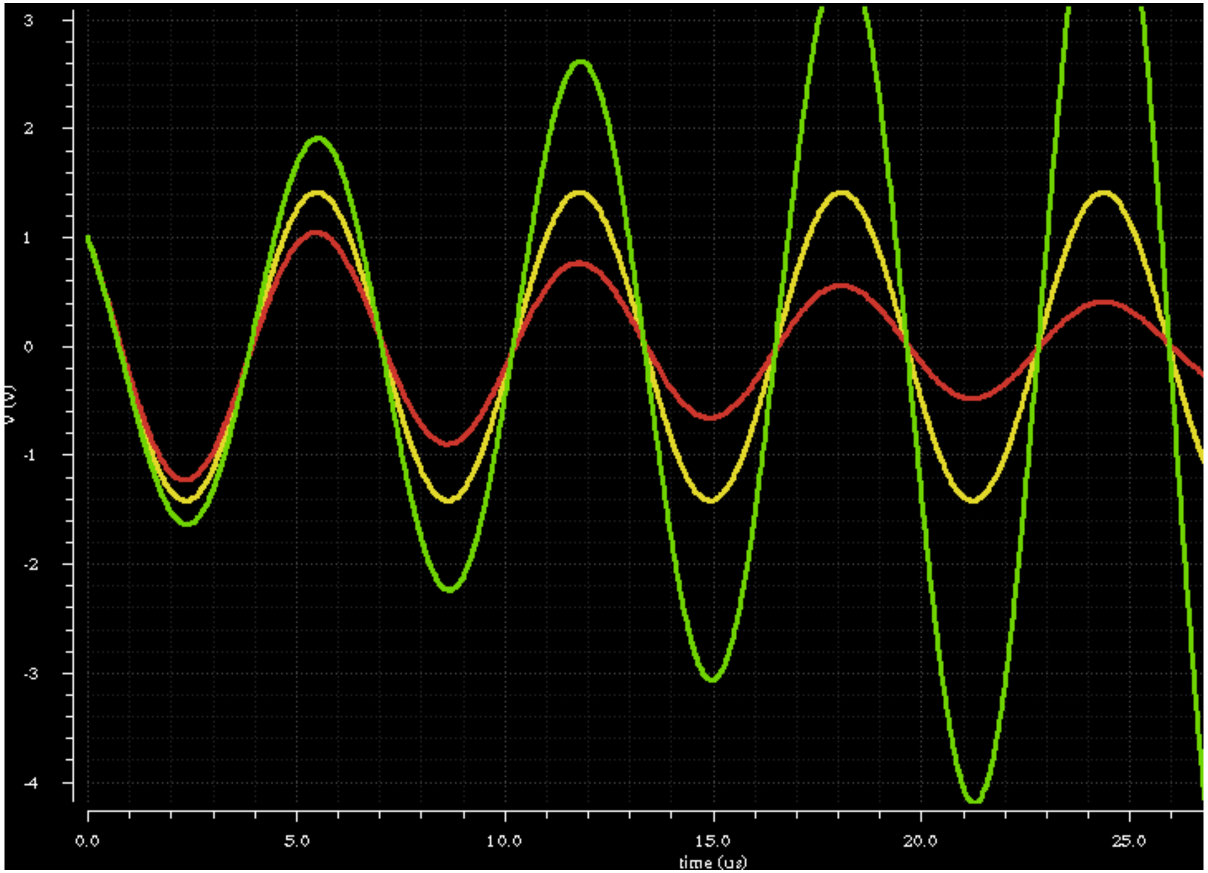
- Centre frequency is at $\omega_0 = 1/RC$.
- Gain there is exactly 1/3.
- Phase is 0 ($H[i\,\omega_0]$ is real:

$$HH\left[\frac{i}{R\,C}\right]$$

$$\frac{1}{3}$$

Solution 5

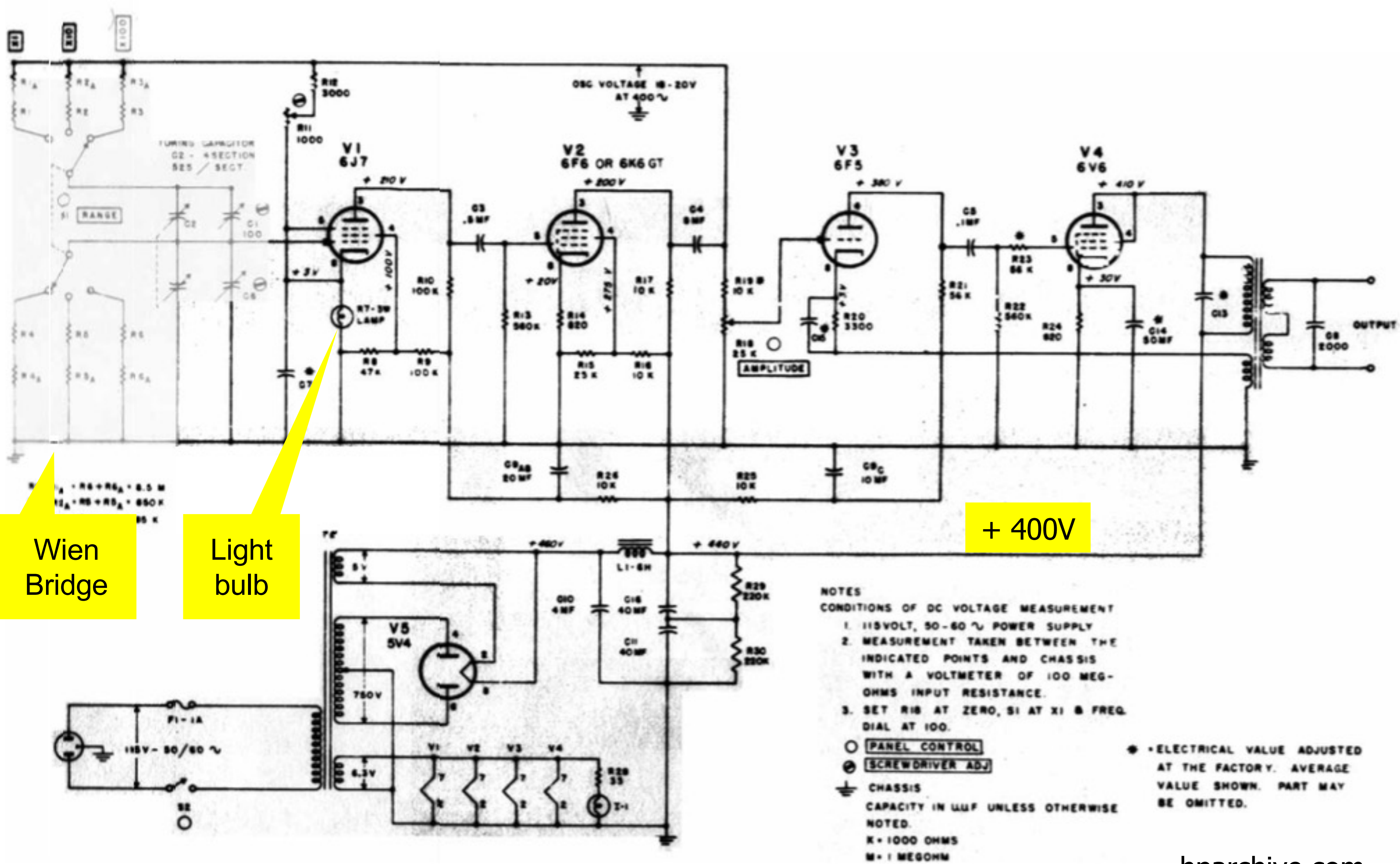
- Oscillator for gain <3,=3,>3:



Itermezzo: Wien Oszillator

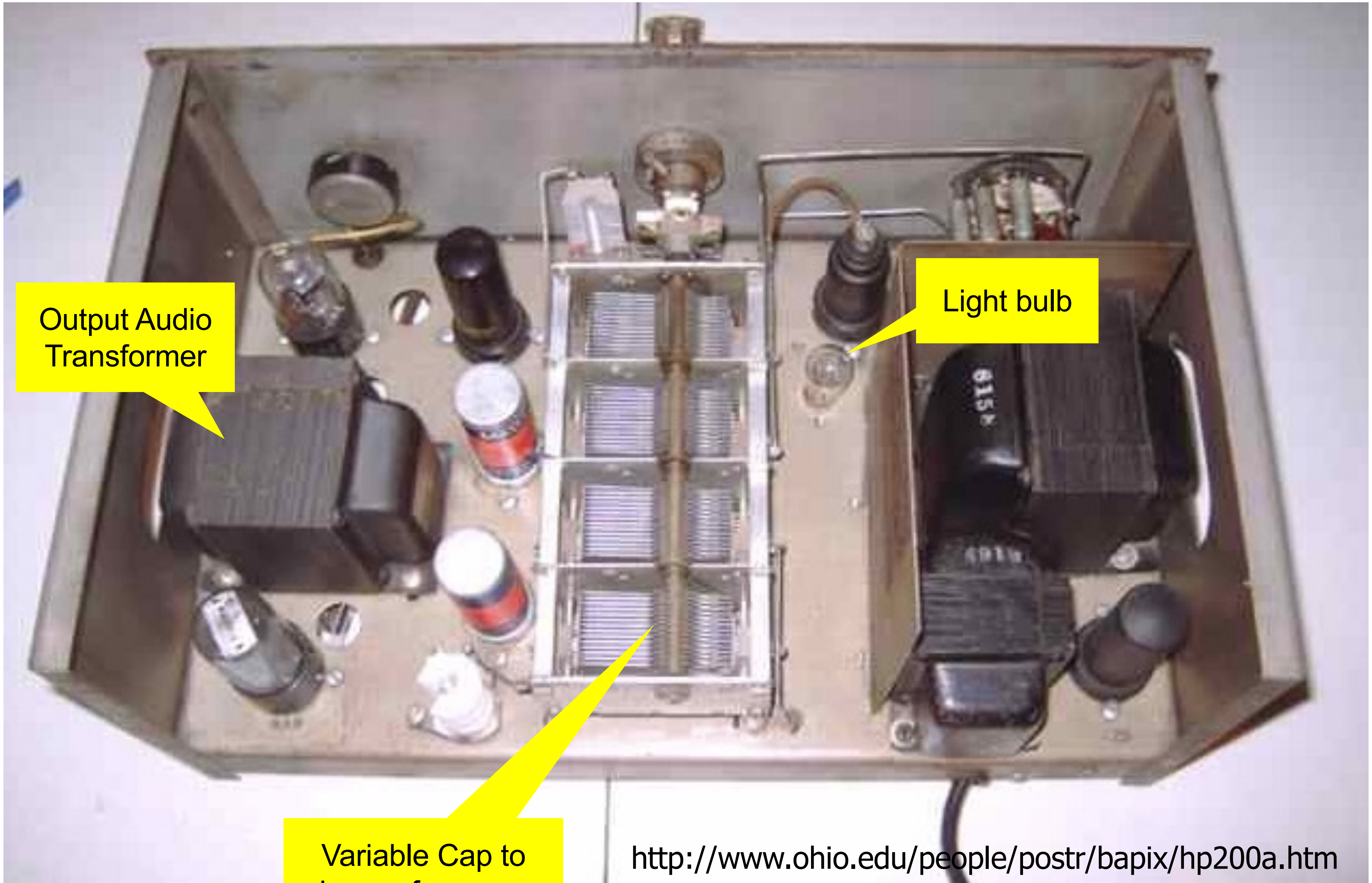
- Wien bridge: Max Wien (1891)
- In 1939, William Hewlett and David Packard (Stanford University) patent an Oscillator using a *light bulb* to stabilize gain (leading to with very low distortion)
- This is the first product of ‘Hewlett – Packard’ (HP): The HP 200A 'Audio Oscillator'

Schematic Diagram (HP 200 B)



hparchive.com

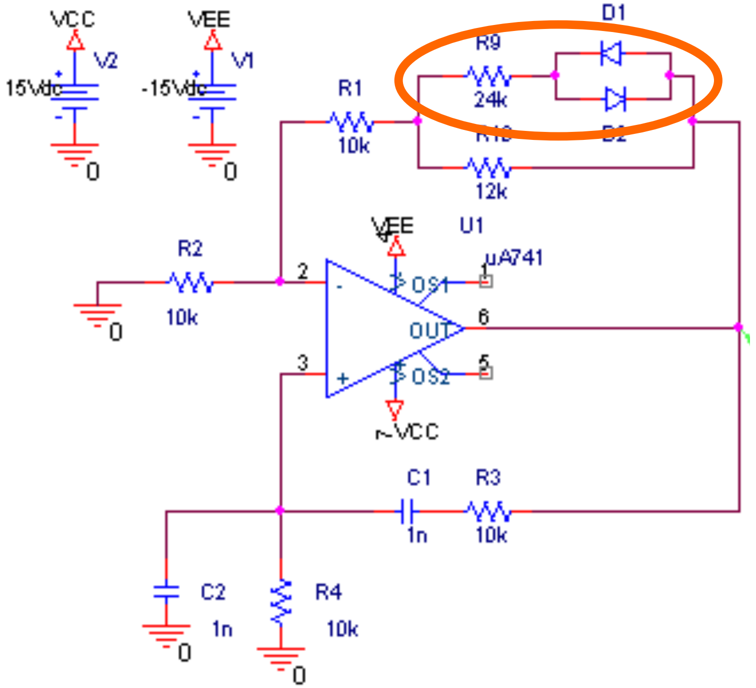
Inside..



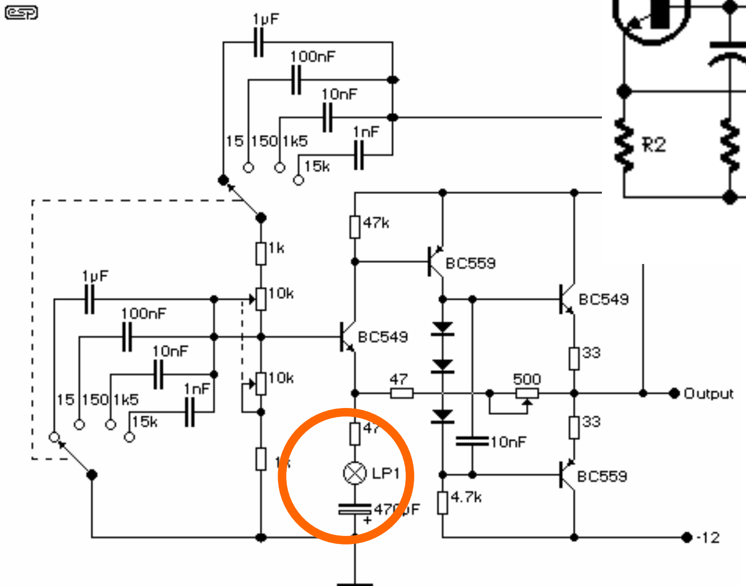
http://www.ohio.edu/people/postr/bapix/hp200a.htm

Discrete circuit versions

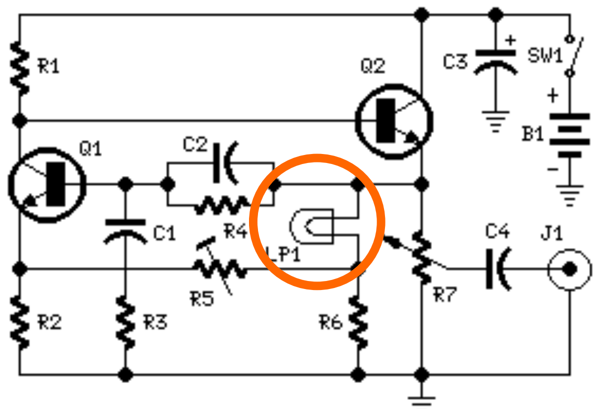
- The problem is to stabilize the gain to exactly 3
- This is achieved by a regulation loop which monitors the output amplitude by some means
- Fast regulation leads to distortion. HP's 'trick' was that the voltage dependent res. of the light bulb varies very slow.



www.calvin.edu/~pribeiro/courses/engr332/Handouts



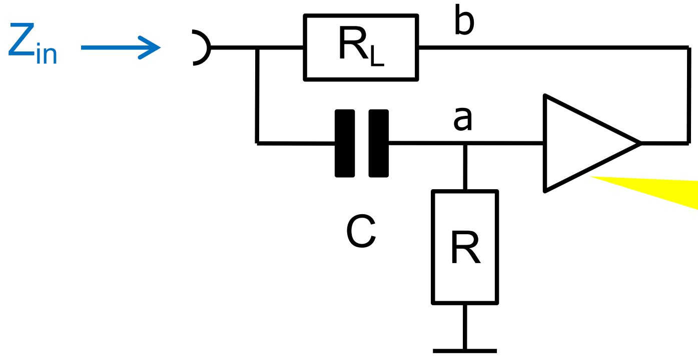
http://sound.westhost.com/project22.htm



http://www.redcircuits.com//Page13.htm

Exercise 6: Gyrator (difficult)

- A 'Gyrator' can mimic inductive behaviour, while using only resistors, capacitors and amplifiers
- Consider the following circuit:



the triangle is a voltage amplifier with gain=1 ('follower').
It forces node b to the potential of node a

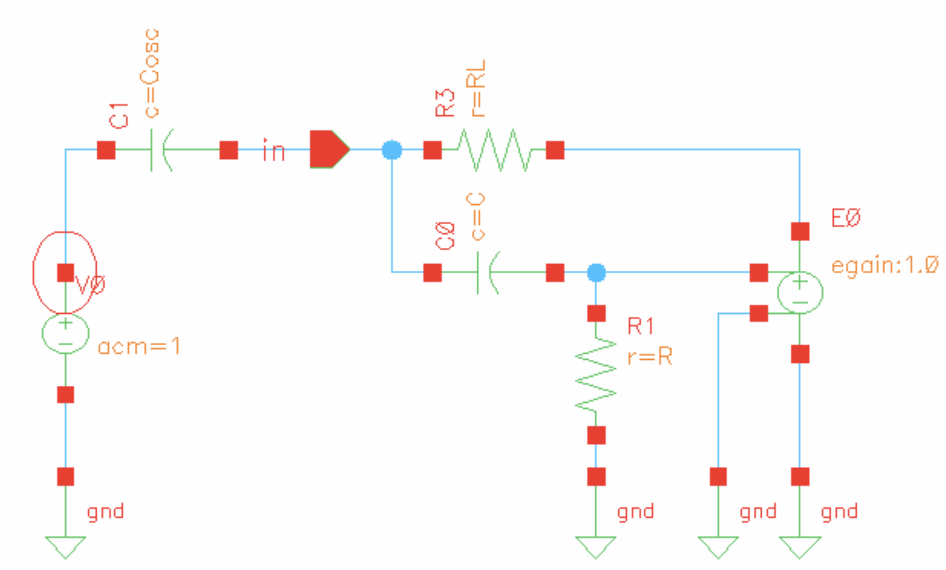
- **Calculate** the input impedance $Z_{in} = U_{in}/I_{in}$ of the circuit
  - (Use Kirchhoff's law at the input node and node a)
- For frequencies < 1/C R_L, the denominator can be neglected.
- Compare the result to an inductor in series with R_L
- Simulate.
  - Note that R should be larger than R_L (what happens for $R=R_L$?)
  - Plot i_{in}.
  - Add another capacitor in series to produce a resonant circuit.

Solution 6

- Mathematica:

```
EQin = iin == (vin - va) s C + (vin - vb)/RL /. vb → va;
EQa = (vin - va) s C == va / R;

Eliminate[{EQin, EQa}, va] // Simplify
iin (RL + C R RL s) == vin + C RL s vin

sol = Solve[%, iin] // First
{iin → (vin + C RL s vin)/(RL (1 + C R s))}

Zgyrator[s_] = vin/iin /. sol // Simplify
(RL + C R RL s)/(1 + C RL s)
```

Numerator is RL + s L with L = C R RL



Simulation with extra Cosc: