Exercise:
Making a Steep Filter

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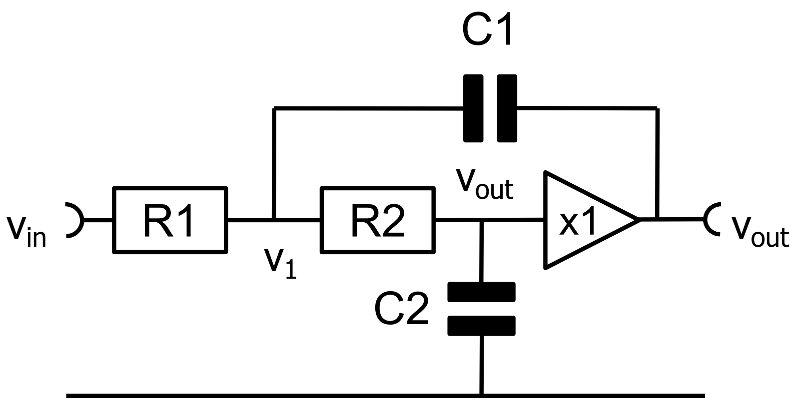
Exercise

- Just for fun, we want to design a steep Butterworth Low Pass filter
  - Let the corner frequency be 1 MHz
  - Let's chose order N=6
  - Implement it using 'Sallen and Key' stages with k=1



- After you have derived component parameters for the 3 filter stages, simulate the result.

Hints

- The poles of a Butterworth filter are placed on the left half circle with equal angles (see slide 'choosing the poles' in the lecture), i.e. with $d\phi = \pi/N$ and $r = \omega$.
- Each complex-conjugate pair of poles is handled by one 2nd order 'Sallen and Key' filter. So we need N/2 stages.
- Each filter (with dc gain 1) has a general transfer function of $1/(1+s/p_a)(1+s/p_b) = 1/(1+as+bs^2)$ where p_a and p_b are the two complex conjugate poles.

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Steps

- Step1:
  - Given a pole pair, we want to know the transfer function
  - Write $p_a = r\,(\,\mathrm{Sin}(\phi)+i\,\mathrm{Cos}(\phi)\,)$, $p_b = \ldots$
  - From p_a and p_b, calculate a, b
- Step2:
  - Our filter has 4 parameters (R1,R2,C1,C2), but its behaviour is described by 2 (e.g. corner, peaking), there are several ways to implement it. For example:
  - Set R1=R2=R and C2 = 1nF. This leaves us with 2 parameters
  - Derive the transfer function of a filter stage
- Step3:
  - For a given (r, ϕ) and thus (a,b), derive R and C1 by equating the coefficients of s and s^2.
- Step 4:
  - Derive (r, ϕ) for each pole-pair of the Butterworth and get R and C1 for that filter stage.

Step 1

```
$Assumptions = r > 0 && ϕ ∈ Reals; (* needed for Conjugate[] *)
```

Assume we have a complex conjugate pole pair (p_a, p_b) with radius r and angles $\pm\phi$:

```
In[8]:= pa = r (Cos[ϕ] + ⅈ Sin[ϕ]); pb = Conjugate[pa] // Simplify; {pa, pb}
Out[8]= {r (Cos[ϕ] + ⅈ Sin[ϕ]), r (Cos[ϕ] - ⅈ Sin[ϕ])}
```

Den = $\left(1 + \frac{s}{p_a}\right)\left(1 + \frac{s}{p_b}\right)$ // Simplify (* This is the denominator of the TF *)

Out[11]= $\frac{r^2 + s^2 + 2\, r\, s \,\mathrm{Cos}[\phi]}{r^2}$

In[13]:= {a, b} = Table[SeriesCoefficient[Den, {s, 0, k}], {k, 1, 2}]

Out[13]= $\left\{\frac{2\,\mathrm{Cos}[\phi]}{r}, \frac{1}{r^2}\right\}$

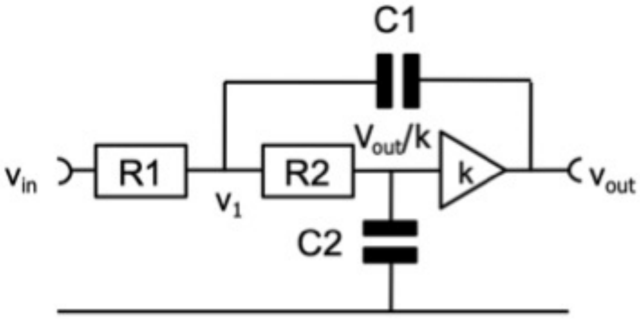
s

s^2

This function finds a Taylor coefficient of function ‘Den’. It assumes variable s, expands at 0 and looks for degree k

Step 2a

- General Derivation of H[s]:



In[39]:= $EQ1 = \frac{vin - v1}{R1} == \frac{v1 - v2}{R2} + (v1 - vout)\, s\, C1;$

$EQ2 = \frac{v1 - v2}{R2} == v2\, s\, C2;$

$EQ3 = vout == k\, v2;$

In[42]:= `Eliminate[{EQ1, EQ2, EQ3}, {v1, v2}] // Simplify`

Out[42]= $k\, vin == (1 + C2\, (R1 + R2)\, s + C1\, R1\, s\, (1 - k + C2\, R2\, s))\, vout$

In[43]:= `Solve[%, vout] // First`

Out[43]= $\left\{vout \to -\frac{k\, vin}{-1 - C1\, R1\, s - C2\, R1\, s + C1\, k\, R1\, s - C2\, R2\, s - C1\, C2\, R1\, R2\, s^2}\right\}$

In[44]:= $H[s\_] = \frac{vout}{vin}$ `/. % /. k → 1 // Simplify`

Out[44]= $\frac{1}{1 + C2\, s\, (R1 + R2 + C1\, R1\, R2\, s)}$

Step 2b and 3

- Now fix R1=R2=RR, C2 = 1nF and C1=CC
  - For some reason using C1 etc. in Mathematica causes trouble.. I have not yet found out why. So I use RR and CC….

In[127]:= Coef = Table$\left[\text{SeriesCoefficient}\left[\frac{1}{\text{H[s] /. }\{\text{R1}\to\text{RR, R2}\to\text{RR, c2}\to 10^{-9}\text{, c1}\to\text{CC}\}}, \{s, 0, kk\}\right], \{kk, 1, 2\}\right]$

Out[127]= $\left\{\frac{\text{RR}}{500\,000\,000}, \frac{\text{CC RR}^2}{1\,000\,000\,000}\right\}$

Fix components in H[s], take denominator, find coefficients of s and s^2

In[128]:= {EQA, EQB} = {Coef[[1]] == a, Coef[[2]] == b} // Simplify

Out[128]= $\{\text{r RR} == 1\,000\,000\,000\,\text{Cos}[\phi], \text{CC r}^2\,\text{RR}^2 == 1\,000\,000\,000\}$

Equate these coefficients with a and b
This gives 2 equations EQA and EQB

In[166]:= Solve[{EQA, EQB}, {RR, CC}] // First

Out[166]= $\left\{\text{RR}\to\frac{1\,000\,000\,000\,\text{Cos}[\phi]}{r}, \text{CC}\to\frac{\text{Sec}[\phi]^2}{1\,000\,000\,000}\right\}$

Solve the 2 equations for RR,CC
(R and CC1 in the exercise)

In[167]:= {Rsol, C1sol} = {RR, CC} /. %

Out[167]= $\left\{\frac{1\,000\,000\,000\,\text{Cos}[\phi]}{r}, \frac{\text{Sec}[\phi]^2}{1\,000\,000\,000}\right\}$

For later: Assign the result to Rsol and Csol

Step 4

This is the angle for one of the two partners of a pair.

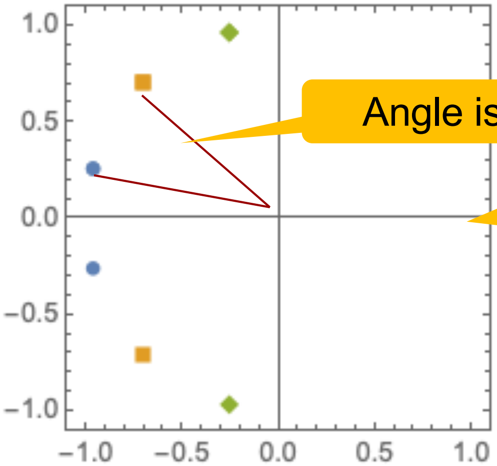
In[77]:= `ORDER = 6;`

In[206]:= ComplexListPlot$\left[\text{Table}\left[-\{p_a, p_b\} \text{ /. } \left\{r \to 1, \phi \to (2\,k - 1)\,\frac{\pi}{2\,\text{ORDER}}\right\}, \left\{k, 1, \frac{\text{ORDER}}{2}\right\}\right]\right.$

`, PlotRange → {{-1.1, 1.1}, {-1.1, 1.1}}`

`, AspectRatio → 1, Frame → True, ImageSize → Small, PlotMarkers → Automatic]`

Out[206]=



Angle is π/ORDER

Illustrate again where the poles are (for r=1)

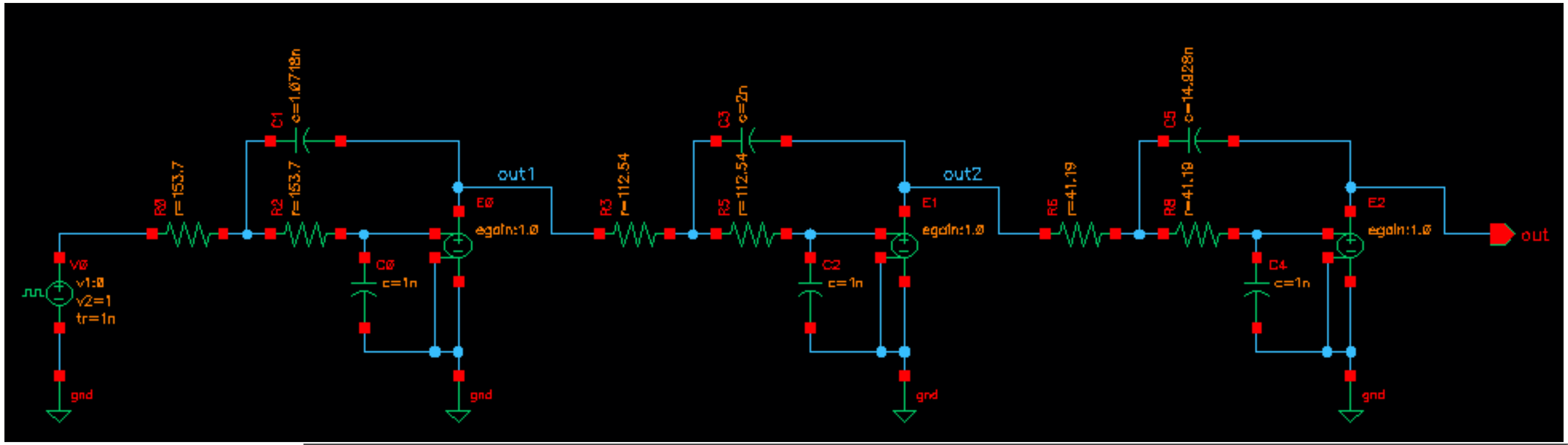
Finally we get Rsol and Csol for the 3 stages!

In[207]:= Table$\left[\left\{\text{Rsol}, \frac{\text{C1sol}}{\text{nF}}\right\} \text{ /. } \left\{r \to 2\,\pi\,10^{6}, \ldots (2\,k - 1)\,\frac{\pi}{2\,\text{ORDER}}\right\}, \left\{k, 1, \frac{\text{ORDER}}{2}\right\}\right]$ `// N`

Out[207]= `{{153.732, 1.0718}, {112.54, 2.}, {41.1923, 14.9282}}`

Simulation

- Schematic:



- AC Sweep: