

4.1 Noise of an RC Low-Pass Filter

We consider a simple RC Low-Pass filter with a resistor R followed by a capacitor C to ground. Even with *no* signal at the input, the thermal noise of the resistor itself leads to noise at the output. This noise is filtered by the Low-Pass so that the total rms noise voltage is finite, despite the infinite white noise spectral density of R . We want to calculate this rms noise and derive an equivalent *noise bandwidth* of the Low-Pass filter. For calculation, perform the following steps:

1. Use a noise *voltage* source in series with R with the correct spectral density $\frac{d\langle n^2 \rangle}{d\nu}$.
2. For a grounded input of the filter, the resistor creates noise. The circuit is equivalent to a (noise) voltage source followed by an /em ideal filter. Write down the (complex) transfer function $H(\omega)$ of the Low-pass filter with corner frequency ω . Derive how the amplitude of the signal is reduced, i.e. $v^2(\omega)$.
3. Integrate the squared noise voltage at the output over all frequencies. Note that you must use $v^2(\omega)$, because we are treating the *squared* noise voltage. Pay attention to integrate over frequency ν , not angular frequency ω . The result is the (squared) rms noise voltage.
4. Do you understand 'physically' why the result is independent of R ?
5. What would be the bandwidth ν_{brick} of an infinitely steep 'brick wall' low-pass filter to obtain the same output noise?
6. Compare this to the corner frequency of the low pass filter (do not confuse ω and ν)!

4.2 High Pass Corner Frequency for Lowest Noise

In the lecture, it was shown how the input referred noise of an amplifier / shaper system can be computed for a $CR^N - RC^M$ shaper. The high- and low-pass sections all had *the same* corner frequencies.

In this (not so simple) exercise, we treat a shaper with *one* low-pass section with corner frequency $\omega_l = 1/\tau_l$ and with *one* high-pass section with *a different* corner frequency $\omega_h = 1/\tau_h$. To simplify expressions, we introduce the ratio between the two frequencies $r := \omega_l/\omega_h = \tau_h/\tau_l$ and describe the system by $\tau_h = \tau$ and r . We want to prove that it is the best choice to have both corner frequencies equal, i.e. to choose $r = 1$. The solution is eased a lot by using a mathematics program like Mathematica.

1. Write down the transfer function $H(s)$ of the system.
2. Calculate the step response (time domain), which is the inverse Laplace Transform $f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} \frac{1}{s} H(s) ds$, where c is chosen to ensure convergence. This integral can be calculated via the residues at the two poles, i.e. the integral is $2\pi i \times \sum \text{Residues}$; the residue of a function $f(x)$ at a simple pole a is $\text{Res}(f(x), a) = \lim_{x \rightarrow a} (x - a) f(x)$.
3. What is the peaking time? Show that the peak amplitude is $r^{\frac{r}{1-r}}$.
4. Verify that for $r \rightarrow 1$ these expressions (step response, peaking time and amplitude) give the values for a normal $CR - RC$ shaper.

5. Assume that the (square) noise voltage at the preamplifier output (i.e. at the input of the shaper) has a spectrum given by $\frac{d\langle v_{pa}^2(\omega) \rangle}{d\omega} = \sum_{k=-2}^0 c_k \omega^k$. Calculate the rms noise at the shaper output by integrating over all frequencies (see page 27 of the lecture slides). Note that the transfer function $H(\omega)$ must be squared! Hint: After a substitution $x = (\omega/\omega_h)^2$, simplify the the integral by expansion into partial fractions. The more complicated of the two resulting integrals can be simplified by substitution and merged with the simpler integral. Solve this by using the 'reflection formula' of the Gamma function.
6. Now divide this result by the (squared) peak amplitude to get the noise referred to a unit signal. In order to do a fair comparison, we need to make sure the systems keeps its speed when we change r . Therefore, choose ω_1 such that the peaking time remains constant. You must probably use a mathematical program to do this now... You can then try to obtain the limit for $r \rightarrow 1$ to verify that you get the result from the lecture slides.
7. The resulting expression is fairly complex and it is difficult to *find* the minimum, but you can verify that the derivative is indeed zero for $r = 1$ for all k *individually*.