4.1 Noise of an RC Low-Pass Filter

We consider a simple RC Low-Pass filter with a resistor R followed by a capacitor C to ground. Even with *no* signal at the input, the thermal noise of the resistor itself leads to noise at the output. This noise is filtered by the Low-Pass so that the total rms noise voltage is finite, despite the infinite white noise spectral density of R. We want to calculate this rms noise and derive an equivalent *noise bandwidth* of the Low-Pass filter. For calculation, perform the following steps:

1. Use a noise *voltage* source in series with R with the correct spectral density $\frac{d\langle n^2\rangle}{d\nu}$.

$$\frac{d\langle n^2\rangle}{d\nu} = 4kT\,R$$

2. For a grounded input of the filter, the resistor creates noise. The circuit is equivalent to a (noise) voltage source followed by an /em ideal filter. Write down the (complex) transfer function $H(\omega)$ of the Low-pass filter with corner frequency ω. Derive how the amplitude of the signal is reduced, i.e. $v^2(\omega)$.

$$H(\omega) = \frac{1}{1 + i\,\omega\tau} \qquad (\tau = RC)$$

$$v^2(\omega) = |H(\omega)|^2 = H(\omega)H^*(\omega) = \frac{1}{(1 + i\,\omega\tau)(1 - i\,\omega\tau)} = \frac{1}{1 + (\omega\tau)^2}$$

3. Integrate the squared noise voltage at the output over all frequencies. Note that you must use $v^2(\omega)$, because we are treating the *squared* noise voltage. Pay attention to integrate over frequency ν, not angular frequency ω. The result is the (suquared) rms noise voltage.

$$rms^2 = \int_0^\infty \frac{4kT\,R}{(1 + (2\pi\nu\tau)^2)} d\nu = \frac{4kT\,R}{2\pi\tau} \int_0^\infty \frac{dx}{1 + x^2} = \frac{kT\,R}{\tau} = \frac{kT}{C}$$

4. Do you understand 'physically' why the result is independent of R?
The magnitude of the white noise voltage increases linearly with R, but in the same time the bandwidth decreases by the same factor as R increases. The two effects just cancel.

5. What would be the bandwidth ν_{brick} of an infinitely steep 'brick wall' low-pass filter to obtain the same output noise?

$$\int_0^{\nu_{brick}} 4kT\,R\,d\nu = 4kT\,R\,\nu_{brick} \stackrel{!}{=} \frac{kT}{C} \Rightarrow \nu_{brick} = \frac{1}{4\,RC}$$

6. Compare this to the corner frequency of the low pass filter (do not confuse ω and ν)!

$$\frac{1}{RC} = \omega_{3dB} = 2\pi\,\nu_{3dB} \Rightarrow \nu_{brick} = \frac{\pi}{2}\,\nu_{3dB} \approx 1.5\,\nu_{3dB}$$

4.2 High Pass Corner Frequency for Lowest Noise

In the lecture, it was shown how the input referred noise of an amplifier / shaper system can be computed for a $CR^N - RC^M$ shaper. The high- and low-pass sections all had *the same* corner frequencies.
In this (not so simple) exercise, we treat a shaper with *one* low-pass section with corner frequency $\omega_\text{l} = 1/\tau_\text{l}$ and with *one* high-pass section with *a different* corner frequency $\omega_\text{h} = 1/\tau_\text{h}$. To simplify expressions, we introduce the ratio between the two frequencies $r := \omega_\text{l}/\omega_\text{h} = \tau_\text{h}/\tau_\text{l}$ and describe the system by $\tau_\text{h} = \tau$ and r. We want to prove that it is the best choice to have both corner frequencies equal, i.e. to choose $r = 1$. The solution is eased a lot by using a mathematics program like Mathematica.

1. Write down the transfer function $H(s)$ of the system.

$$H(s) = \frac{s\tau_\text{h}}{1+s\tau_\text{h}}\frac{1}{1+s\tau_\text{l}} = \frac{s\tau_\text{h}}{1+s\tau_\text{h}}\frac{1}{1+s\tau_\text{l}} = \frac{s\tau}{(s\tau+1)\left(\frac{s\tau}{r}+1\right)}.$$

2. Calculate the step response (time domain), which is the inverse Laplace Transform $f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\frac{1}{s}H(s)ds$, where c is chosen to ensure convergence. This integral can be calculated via the residues at the two poles, i.e. the integral is $2\pi i \times \sum$ Residues; the residue of a function $f(x)$ at a simple pole a is $Res(f(x), a) = \lim_{x\to a}(x-a)f(x)$.
We must integrate

$$f(s) = \frac{\tau_\text{h}\, e^{st}}{(1+s\tau_\text{h})(1+s\tau_\text{l})}$$

along a vertical line in the imaginary plane. The poles of $f(s)$ are $p_l = -1/\tau_\text{l}$ and $p_h = -1/\tau_\text{h}$ with the corresponding residuae

$$\begin{aligned}\text{Res}(p_l) &= \lim_{s\to-\frac{1}{\tau_\text{l}}}\left(s+\frac{1}{\tau_\text{l}}\right)e^{st}\frac{\tau_\text{h}}{1+s\tau_\text{h}}\frac{1}{1+s\tau_\text{l}} = \lim_{s\to-\frac{1}{\tau_\text{l}}}\frac{\tau_\text{h}}{\tau_\text{l}}\frac{e^{st}}{1+s\tau_\text{h}} = \tau_\text{h}\frac{e^{-\frac{t}{\tau_\text{l}}}}{\tau_\text{l}-\tau_\text{h}}\\ \text{Res}(p_h) &= \tau_\text{h}\frac{e^{-\frac{t}{\tau_\text{h}}}}{\tau_\text{h}-\tau_\text{l}}.\end{aligned}$$

The step response $f(t)$ is the sum of these two values (the factors $2\pi i$ in the formula for the inverse Laplace transform and for the line integral just cancel out!):

$$f(t) = \tau_\text{h}\frac{e^{-\frac{t}{\tau_\text{h}}} - e^{-\frac{t}{\tau_\text{l}}}}{\tau_\text{h}-\tau_\text{l}} = \frac{r}{1-r}\left(e^{-r\frac{t}{\tau_\text{h}}} - e^{-\frac{t}{\tau_\text{h}}}\right). \tag{1}$$

3. What is the peaking time? Show that the peak amplitude is $r^{\frac{r}{1-r}}$.
The peaking time is obtained by setting the derivative (with respect to t)

$$f'(t) = \frac{r}{1-r}\frac{1}{\tau_\text{h}}\left(e^{-\frac{t}{\tau_\text{h}}} - re^{-\frac{t}{\tau_\text{l}}}\right)$$

to zero. We get

$$t_{peak} = \tau_\text{h}\frac{\ln r}{r-1}.$$

The peak amplitude is

$$f_{max}(r) = r^{\frac{1}{1-r}}.$$

4. Verify that for $r \to 1$ these expressions (step response, peaking time and amplitude) give the values for a normal $CR - RC$ shaper.
   The best method to find the limits for $r \to 1$ is to use the *rule of de l'Hospital*. This states that for a fraction in which both, numerator and denominator, tend to zero, the limit can be obtained by taking the derivatives in both parts.

   For the step response, taking the derivative with respect to r leads to

   $$\lim_{r\to 1} f(t) = \frac{t}{\tau_\mathrm{h}} e^{-\frac{t}{\tau_\mathrm{h}}}$$

   as expected.

   The limit of t_{peak} becomes:

   $$\lim_{r\to 1} \tau_\mathrm{h} \frac{\ln r}{r-1} = \lim_{r\to 1} \tau_\mathrm{h} \frac{1/r}{1} = \tau_\mathrm{h}.$$

   The limit of $f_{max}(r)$ is a bit more difficult to find. A possible trick is to first calculate the logarithm

   $$g(r) = \log[f_{max}(r)] = \frac{\log(r)}{1-r}.$$

   The limit of $g(r)$ for $g \to 1$ can now be obtained again using de l'Hospital:

   $$\lim_{r\to 1} g(r) = \lim_{r\to 1} \frac{\log(r)'}{(1-r)'} = \lim_{r\to 1} \frac{1/r}{-1} = -1.$$

   As $f_{max}(r) = e^{g(r)}$, we have $\lim_{r\to 1} f_{max}(r) = 1/e$ as expected.

5. Assume that the (square) noise voltage at the preamplifier output (i.e. at the input of the shaper) has a spectrum given by $\frac{d\langle v_{pa}^2(\omega)\rangle}{d\omega} = \sum_{k=-2}^{0} c_k \omega^k$. Calculate the rms noise at the shaper output by integrating over all frequencies (see page 27 of the lecture slides). Note that the transfer function $H(\omega)$ must be squared! Hint: After a substitution $x = (\omega/\omega_\mathrm{h})^2$, simplify the the integral by expansion into partial fractions. The more complicated of the two resulting integrals can be simplified by substitution and merged with the simpler integral. Solve this by using the 'reflection formula' of the Gamma function.
   First we need the absolute value of the square of the transfer function

   $$|H(i\omega)|^2 \;=\; H(i\omega)\cdot H^*(i\omega) = \frac{r^2\omega^2{\omega_\mathrm{h}}^2}{({\omega_\mathrm{h}}^2+\omega^2)\,(r^2{\omega_\mathrm{h}}^2+\omega^2)}$$

   The integral over the noise spectrum is

   $$\begin{aligned} v_{sha}^2 &= \sum_{k=-2}^{0} c_k r^2 \int_0^\infty \frac{\omega^{k+2}{\omega_\mathrm{h}}^2 \mathrm{d}\omega}{({\omega_\mathrm{h}}^2+\omega^2)\,(r^2{\omega_\mathrm{h}}^2+\omega^2)} \\ &= \ldots \end{aligned}$$

and after the above steps

$$v_{sha}^2 = \frac{c_k \pi {\omega_\mathrm{h}}^{k+1}}{2\cos\left(\frac{\pi k}{2}\right)} \frac{r^2\left(r^{k+1}-1\right)}{r^2-1}$$

for each of the three components ($k = -2 \ldots 0$).

6. Now divide this result by the (squared) peak amplitude to get the noise referred to a unit signal. In order to do a fair comparison, we need to make sure the systems keeps its speed when we change r. Therefore, choose ω_l such that the peaking time remains constant. You must probably use a mathematical program to do this now... You can then try to obtain the limit for $r \to 1$ to verify that you get the result from the lecture slides.

   This becomes quite lengthy be hand, so I used a mathematical program to avoid mistakes. We first must divide the previous result by the (squared) peak amplitude $f_{max}^2(r) = r^{\frac{2}{1-r}}$. The peaking time of the filter is higher by a factor $\gamma = \frac{\ln r}{r-1}$ compared to the simple CR-RC filter with equal corner frequencies. To compensate for this, we must *divide* our peaking time by γ or *multiply* the corner *frequency* by γ. Putting this together, we obtain

$$\langle v^2 \rangle = \frac{\pi c_k \, r^{\frac{2r}{r-1}} \left(r^{k+1}-1\right)\left(\frac{\omega_{CRRC}\log(r)}{r-1}\right)^{k+1}}{2\left(r^2-1\right)\cos\left(\frac{\pi k}{2}\right)}$$

   for each of the three components ($k = -2 \ldots 0$).

   The limit for $r \to 1$ is
$$\frac{1}{4} e^2 \pi c_k (k+1) \omega_{CRRC}^{k+1} \sec\left(\frac{\pi k}{2}\right).$$
   Setting $k = -2, -1, 0$ yield the result of the lecture, as required.

7. The resulting expression is fairly complex and it is difficult to *find* the minimum, but you can verify that the derivative is indeed zero for $r = 1$ for all k *individually*.
   Let you math program calculate there derivative wrt. r and determine the limit of $k \to 1$. All 3 derivatives are zero indeed.